2 $p$thly payments, annuities, loans and funds

Please hand in your answers to questions 1, 4, 5, 6, 9 and 12 for marking. The rest are for further practice.

1. For $m \in \mathbb{N}$, the prefix $_{m|}$ before an annuity symbol indicates that the sequence of payments concerned is *deferred* by an amount of time m. For example, the discounted present value (in the constant interest-rate model) of a *deferred annuity*, with unit payments per unit time payable from $m+1$ to $m+n$, is denoted by $_{m|}a_{\overline{n}|}$.

   (a) Express $_{m|}a_{\overline{n}|}$ in terms of the ordinary annuity symbols introduced in the lectures.

   (b) The case $m = -1$ corresponds to an annuity-due and is denoted by $\ddot{a}_{\overline{n}|}$. Express $\ddot{a}_{\overline{n}|}$ as simply as you can (i) in terms of $a_{\overline{n}|}$ and (ii) in terms of $a_{\overline{n-1}|}$

   (c) The discounted present value of an *increasing annuity* with payments j at time $j = 1, \ldots, n$ is denoted by $(Ia)_{\overline{n}|}$. Express $(Ia)_{\overline{n}|}$ in terms of ordinary annuity symbols.

   (d) Consider a security redeemable at par, with term n and $p$thly coupon payments at nominal rate j. Express the accumulated (time n) and discounted (time 0) values in the constant i model in terms of annuity symbols.

2. Find, on the basis of an effective interest rate of 4% per unit time, the values of

$$a^{(4)}_{\overline{67}|}, \quad s^{(12)}_{\overline{18}|}, \quad \ddot{a}^{(4)}_{\overline{16.5}|}, \quad \ddot{s}^{(12)}_{\overline{15.25}|}, \quad {}_{4.25|}a^{(4)}_{\overline{3.75}|}, \quad (Is)_{\overline{4}|}.$$

   Describe the meaning of each of the symbols.

3. Buy now, pay later!! A luxury sofa is sold with the following payment options: either pay immediately and receive a 5% discount, or pay by monthly instalments (in arrears) for 15 months. To calculate the monthly payments, add a 5% finance charge to the purchase price, then divide the total amount into 15 equal payments. What annual effective rate of interest is being charged for paying by instalments?

4. Show that in any constant i interest rate model, the cash-flows $c_\infty(s) = \delta$, $0 \le s \le 1$, and $c_p = (k/p, i^{(p)}/p)_{k=1,\ldots,p}$ are equivalent (have the same value) for all $p \in \mathbb{N}$.

5. A loan of £30,000 is to be repaid by a level annuity payable monthly in arrears for 25 years, and calculated on the basis of an (effective) interest rate of 12% pa. Calculate the intitial monthly repayments.

   After ten years of repayments the borrower asks to:

   (a) pay off the loan which is outstanding. Calculate the lump sum which would be required to pay off the outstanding loan.

   (b) extend the loan by a further five years (i.e. to 30 years in total), and with repayments changed from monthly to quarterly in arrears. Calculate the revised level of quarterly repayments.

   (c) reduce the loan period by five years (i.e. to 20 years in total) and for repayments to be biannually (i.e. once every two years) in arrears. Calculate the revised level biannual repayments.

   [*Hint: calculate annual repayment levels first, then combine each two consecutive payments into an equivalent single payment.*]

6. An insurance company issues an annuity of £10,000 p.a. payable monthly in arrears for 25 years. The cost of the annuity is calculated using an effective rate of 10% p.a.
   (a) Calculate the interest component of the first instalment of the sixth year.
   (b) Calculate the total interest paid in the first 5 years.

7. A 10-year loan for £10,000 has a fixed rate of 5% for the first 3 years. Thereafter the rate is 8% for the rest of the term. Repayments are annual. Find the amounts (i) of the first 3 payments and (ii) of the remaining 7 payments. Show that the APR of the loan is 6.4%.

8. (a) An annuity-certain is payable annually in advance for n years. The first payment of the annuity is 1. Thereafter the amount of each payment is $(1+r)$ times that of the preceding payment. Show that, on the basis of an interest rate of i per annum, the present value of the annuity is $\ddot{a}_{\overline{n}|j}$ where $j = (i-r)/(1+r)$.
   (b) Suppose instead that the annuity is payable annually in arrear. Is its present value (at rate i) now equal to $a_{\overline{n}|j}$?
   (c) In return for a single premium of £10,000 an investor will receive an annuity payable annually in arrear for 20 years. The annuity payments increase from year to year at the (compound) rate of 5% per annum. Given that the initial amount of the annuity is determined on the basis of an interest rate of 9% per annum, find the amount of the first payment.

9. A couple buying a house require a £150,000 mortgage. They know that they will have to sell the house (and redeem the mortgage) in three years' time. They have the choice of two mortgages with monthly payments and 25-year term.

   Mortgage A is at effective rate $i_A = 10.25\%$ p.a. This mortgage stipulates, however, that if you pay off the mortgage any time before the fifth anniversary, you will have to pay a penalty equal to 2.4% of the outstanding debt at the time of repayment.

   Mortgage B is at $i_B = 10.7\%$ but can be paid off at any time without penalty.

   Given that the buyers will have to repay the mortgage after three years and that they can save money at $j = 6\%$, which mortgage should they choose?

   Calculate the yields (assuming redemption after three years).

10. *Hardy's Formula.* Consider a fund F, and incomplete information as follows: the amounts of the fund $F_0 \in \mathbb{R}_+$ just before time 0 and $F_1 \in \mathbb{R}_+$ just after time 1 and the (net) total new money $N \in \mathbb{R}$ invested during $[0, 1]$. Denote $I = F_1 - F_0 - N$, the total interest and capital gain earned by the fund during $[0, 1]$.

    (a) Assume that N is received in two equal instalments at the beginning and end of the year. Show that the money-weighted rate of return is given by the formula
    $$i = \frac{2I}{F_1 + F_0 - I}.$$
    What is the time-weighted rate of return?
    (b) Assume that N is received in one instalment at $t = 1/2$. Find an expression for the MWRR. What about the TWRR?
    (c) Assume that N is received continuously at constant rate in $[0, 1]$. Give the yield equation for the money-weighted rate of return i. Show that for small i, the formula in part (a) is still a good approximation for the MWRR.

11. (i) A fund had the following revenue account for 2004:

		£million
Value of fund at 1 January 2004		30
Add: new investments received during the year	18	
Deduct: withdrawals and other payments	30	
Value of fund at 31 December 2004		21

Assuming the new investments received and the withdrawals and other payments are spread evenly over the year, use Hardy's Formula from the previous question to caluclate an approximate effective annual rate of interest for 2004.

(ii) A second set of accounts was constructed to show the dates at which the new investments were received, the dates at which the withdrawals and other payments actually occurred, and the value of the fund at various times during the year:

		£million
Value of fund at 1 January 2004		30
Value of fund at 31 March		36
Value of fund at 15 May		35
New investments at 16 May	18	
Value of fund at 30 June		51
Value of fund at 30 September		45
Withdrawals on 1 October	30	
Value of fund at 31 December 2004		21

(a) Show that the annual yield (i.e. the annual money weighted rate of return) on the fund for the year ending 31 December 2004 is approximately 8.87%.

(b) Calculate the annual TWRR on the fund for the year ending 31 Dec 2004.

(iii) Explain the differences between your answers for (i), (ii)(a) and (ii)(b).

12. In a particular accumulation fund income is retained and used to increase the value of the fund unit. The 'middle price' of the unit on 1 April in each of the years 1999 to 2005 is given in the following table:

Year 1 April	1999	2000	2001	2002	2003	2004	2005
Middle price of unit in £	1.86	2.11	2.55	2.49	2.88	3.18	3.52

(a) On the basis of the above prices and ignoring taxation and expenses:

(i) Find the time-weighted rate of return over the period 1 April 1999 to 1 April 2005,

(ii) Show that the yield obtained by an investor who purchased 200 units on 1 April in each year from 1999 to 2004 inclusive, and who sold his holding on 1 April 2005, is approximately 10.60%.

(iii) Show that the yield obtained by a person who invested £500 in the fund on 1 April each year from 1999 to 2004 inclusive, and who sold back his holding to the fund managers on 1 April 2005, is approximately 10.67% (You should assume that investors may purchase fractional parts of units.)

(b) Suppose that, in order to allow for expenses, the fund's managers sell units 2% above the published middle price and buy back units 2% below the middle price. On this basis find revised answers to (ii) and (iii) of (a).