B.2 Poisson counting measures

1. (a) The distribution of Π is specified in terms of the associated counting measure

$$N((a,b]) = \#\Pi \cap (a,b] = \{j \geq 1 : a < T_j \leq b\} = X_b - X_a, \qquad 0 \leq a < b.$$

Clearly, N satisfies property $^{\text{hom}}$(b) of a Poisson counting measure: $N((a,b]) = X_b - X_a \sim \text{Poi}(\lambda(b-a))$ by the stationarity (ii) and Poisson (iv) properties for increments of X, and we identify the constant intensity function $\lambda(t) = \lambda$, $t \geq 0$.

N also satisfies (a), since for disjoint intervals $(a_j, b_j]$, $j = 1, \ldots, n$, we have $N((a_j, b_j]) = X_{b_j} - X_{a_j}$ increments of X over disjoint time intervals. By property (i) of the Poisson process, these are independent, as required.

(b) (i) Let $0 \leq t_0 < t_1 < \ldots < t_n$. Since N is a measure, we have $N((t_{j-1}, t_j]) = N([0,t_j]) - N([0,t_{j-1}]) = X_{t_j} - X_{t_{j-1}}$. Since the sets $A_j = (t_{j-1}, t_j]$, $j = 1, \ldots, n$, are disjoint, property (a) of the Poisson counting measure yields the independence of the increments of X.

(ii) Fix $r \geq 0$. For an increment $X_{s+r} - X_s = N((s, s+r])$, property $^{\text{inhom}}$(b) of the Poisson counting measure yields a Poisson distribution with parameter $p_r(s) = \int_s^{s+r} \lambda(x)dx$. The differentiable function $s \mapsto p_r(s)$ is constant if and only if $0 = p_r'(s) = \lambda(s+r) - \lambda(s)$ for all $s \geq 0$.
Now $(X_t)_{t \geq 0}$ has stationary increments if and only if $s \mapsto p_r(s)$ is constant for all $r \geq 0$ if and only if $\lambda(s) = \lambda(r+s)$ for all $r \geq 0$, $s \geq 0$. This is the case if and only if $x \mapsto \lambda(x)$ is constant.

(iii) Clearly $t \mapsto X_t$ is an increasing function, so all left and right limits exists. Denote by Π the associated spatial Poisson process, then $\Pi = \{t \geq 0 : \Delta X_t > 0\} = \{t \geq 0 : \Delta X_t = 1\}$. The set Π cannot have accumulation points since λ is locally integrable, so $\Pi = \{T_j, j \geq 1\}$ and $X_t = N([0,t]) = j$ for $t \in [T_j, T_{j+1})$ is right-continuous at jump times, continuous elsewhere.

(iv) $X_t - X_s = N((s,t]) = \text{Poi}(\int_s^t \lambda(x)dx)$, by property $^{\text{inhom}}$(b) of the Poisson counting measure.

(v) $\mathbb{P}(T_1 > s) = \mathbb{P}(N([0,s]) = 0) = \exp\{-\int_0^s \lambda(x)dx\}$ for all $s \geq 0$.

(vi) The density of T_1 is obtained by differentiating the survival function:

$$f_{T_1}(s) = \lambda(s) \exp\left\{-\int_0^s \lambda(x)dx\right\}.$$

To calculate the joint distribution of $(T_1, T_2 - T_1)$, first calculate the joint distribution of (T_1, T_2), from

$$\begin{aligned}\mathbb{P}(T_1 > s, T_2 > t) &= \mathbb{P}(N([0,s]) = 0, N((s,t]) \leq 1) \\ &= \exp\left\{-\int_0^s \lambda(x)dx\right\}\left(1 + \int_s^t \lambda(x)dx\right)\exp\left\{-\int_s^t \lambda(x)dx\right\}\end{aligned}$$

and differentiation, first with respect to s then with respect to t

$$f_{T_1,T_2}(s,t) = \lambda(s)\lambda(t)\exp\left\{-\int_0^t \lambda(x)dx\right\}$$

and the transformation formula for $(T_1, T_2) \mapsto (T_1, T_2 - T_1)$ gives

$$f_{T_1,T_2-T_1}(s,r) = \lambda(s)\lambda(s+r)\exp\left\{-\int_0^{s+r}\lambda(x)dx\right\}$$

and then

$$\begin{aligned} f_{T_2-T_1|T_1=s}(r) &= \lambda(s+r)\exp\left\{-\int_s^{s+r}\lambda(x)dx\right\} \\ &\Rightarrow \mathbb{P}(T_2 - T_1 > r|T_1 = s) = \exp\left\{-\int_s^{s+r}\lambda(x)dx\right\} \end{aligned}$$

is independent of s for all $r \geq 0$ if and only $x \mapsto \lambda(x)$ is constant, by the argument given in (ii).

2. (a) Fix $\beta > 0$. Note that the formula reduces to $0 = 0$ for $\gamma = 0$. It is therefore sufficient to show that the γ-derivatives of both sides coincide. To differentiate the left hand side, note that

$$\frac{\partial}{\partial\gamma}(e^{\gamma x} - 1)\frac{1}{x}e^{-\beta x} = e^{\gamma x}e^{-\beta x} \leq e^{-\beta x},$$

for $\gamma \leq 0$, where $x \mapsto e^{-\beta x}$ is integrable on $[0, \infty)$. Therefore, we may interchange γ-differentiation and x-integration and have to show that for all $\gamma < 0$

$$\int_0^\infty e^{\gamma x}e^{-\beta x}dx = \frac{1}{1-\gamma/\beta}\frac{1}{\beta},$$

which clearly is true.

The argument works for $\gamma \leq \gamma_0$ if we choose $e^{-(\beta-\gamma_0)}$ as integrable upper bound. Clearly, for every fixed $\gamma < \beta$, any $\gamma_0 \in (\gamma, \beta)$ will do.

(b) We apply the exponential formula for Poisson point processes and (a) to obtain

$$\mathbb{E}\left(\exp\left\{\gamma\sum_{s\leq t}\Delta_s\right\}\right) = \exp\left\{t\int_0^\infty(e^{\gamma x} - 1)\alpha x^{-1}e^{-\beta x}dx\right\} = \left(\frac{\beta}{\beta-\gamma}\right)^{\alpha t}.$$

We recognise the last expression as the moment generating function of the Gamma distribution with the required density. By the Uniqueness Theorem for moment generating functions, $\sum_{s\leq t}\Delta_s$ has this Gamma distribution.

(c) Fix $0 \leq t_0 < t_1 < \ldots < t_n$. Since $(\Delta_s)_{s\geq 0}$ is a Poisson point process, the processes $(\Delta_s)_{t_{j-1}<s\leq t_j}$, $j = 1, \ldots, n$, are independent (consider the restrictions to disjoint domains $(t_{j-1}, t_j] \times (0, \infty)$ of the Poisson counting measure

$$N((a, b] \times (c, d]) = \{a < t \leq b : \Delta_t \in (c, d]\}, \quad 0 \leq a < b, 0 < c < d),$$

and so are the sums $\sum_{t_{j-1}<s\leq t_j}\Delta_s$ as functions of independent random variables. Fix $s < t$. Then the process $(\Delta_{s+r})_{r\geq 0}$ has the same distribution as $(\Delta_s)_{s\geq 0}$. In particular, $\sum_{0\leq r\leq t}\Delta_{s+t} \sim \sum_{0\leq r\leq t}\Delta_r$. The process $t \mapsto \sum_{s\leq t}\Delta_s$

is right-continuous with left limits, since it is a random increasing function where for each jump time T, we have (by monotone convergence)

$$\lim_{t\uparrow T}\sum_{s\le t}\Delta_s=\sum_{s<T}\Delta_s\quad\text{and}\quad\lim_{t\downarrow T}\sum_{s\le t}\Delta_s=\sum_{s\le T}\Delta_s.$$

3. (a) Denote by $T_n\sim\text{Gamma}(n,\lambda_X)$ and $T'_m\sim\text{Gamma}(m,\lambda_Y)$ the jump times of X and Y. These are independent continuously distributed random variables and so $\mathbb{P}(T_n=T'_m)=0$. Therefore, (by subadditivity)

$$\mathbb{P}(\{T_n,n\ge 1\}\cap\{T'_m,m\ge 1\}\neq\varnothing)\le\sum_{m\ge 1}\sum_{n\ge 1}\mathbb{P}(T_n=T'_m)=0.$$

(b) Denote the Poisson arrival processes of jumps by R^X and R^Y. Then R^X+R^Y satisfies the four properties of the Poisson process, since (i) $R^X_{t_j}-R^X_{t_{j-1}}+R^Y_{t_j}-R^Y_{t_{j-1}}$, $j=1,\ldots,n$, are independent as sums of independent random variables, (ii)/(iv) their distributions are $\text{Poi}(\lambda_X(t_j-t_{j-1})+\lambda_Y(t_j-t_{j-1})$ as sum of two independent Poisson variables only depending on t_j-t_{j-1}, (iii) paths are right-continuous with left limits as sums of two such paths.

(c) We condition on whether $T_1<T'_1$ or $T'_1<T_1$ and get for the first jump size J^D_1 of D

$$\begin{aligned}\mathbb{P}(J^D_1\in A)&=\mathbb{P}(T_1<T'_1)\mathbb{P}(J^X_1\in A|T_1<T'_1)+\mathbb{P}(T_1>T'_1)\mathbb{P}(-J^Y_1\in A)\\&=\frac{\lambda_X}{\lambda_X+\lambda_Y}\mathbb{P}(J^X_1\in A)+\frac{\lambda_Y}{\lambda_X+\lambda_Y}\mathbb{P}(-J^Y_1\in A).\end{aligned}$$

This is a mixture of the jump size distributions of X and Y. We deduce that the density is

$$h_D(x)=\frac{\lambda_X}{\lambda_X+\lambda_Y}h_X(x)+\frac{\lambda_Y}{\lambda_X+\lambda_Y}h_Y(-x)=\begin{cases}\frac{\lambda_X}{\lambda_X+\lambda_Y}h_X(x) & x>0\\ \frac{\lambda_Y}{\lambda_X+\lambda_Y}h_Y(-x) & x<0\end{cases}$$

4. (a) This is bookwork, see Lecture 3, Example 18. The intensity function is $\lambda_X h_X(x)$, $x>0$.

(b) By the previous part, we have two Poisson point process Δ^X and Δ^Y in $(0,\infty)$. It is easy to see that $\Delta^{-Y}=-\Delta_Y$ is a Poisson point process in $(-\infty,0)$ with intensity function $\lambda_Y h_Y(-x)$, $x<0$. It is easy to see that the associated Poisson counting measures on $[0,\infty)\times(0,\infty)$ and $[0,\infty)\times(-\infty,0)$ together form a Poisson counting measure on $[0,\infty)\times\mathbb{R}\setminus\{0\}$ via

$$N(A\times B)=N_X(A\times(B\cap(0,\infty)))+N_Y(A\times(B\cap(-\infty,0))).$$

The intensity function is $\lambda_X h_X(x)$, $x>0$ and $\lambda_Y h_Y(-x)$, $x<0$.

(c) Since $(\Delta D_t)_{t\ge 0}$ is a Poisson point process with integrable intensity function $\int_0^\infty\lambda_X h_X(x)dx+\int_{-\infty}^0\lambda_Y h_Y(-x)dx=\lambda_X+\lambda_Y<\infty$, and $D_t=\sum_{s\le t}\Delta D_s$, D is a compound Poisson process.

(d) For every real-valued compound Poisson process C we can define the processes X and Y of positive and negative jumps. Since the associated processes $(\Delta X_t)_{t\geq 0}$ and $(\Delta Y_t)_{t\geq 0}$ inherit the properties of Poisson point processes (via their Poisson counting measures), this provides the required decomposition into two independent increasing compound Poisson processes. It is unique because any other decomposition must have more jumps, which must happen at the same time and cancel each other, but by (a), this is incompatible with independence.