A.6 Time change

1. Consider Brownian motion \((B_t)_{t \geq 0}\) and a continuous increasing function \(f : [0, \infty) \to [0, \infty)\) with \(f(0) = 0\). Set \(Z_y = B_{f(y)}, y \geq 0\).

   (a) Show that \(Z\) has quadratic variation
   
   \[ [Z]_y := \lim_{n \to \infty} \sum_{j=1}^{\lfloor 2^ny \rfloor} (Z_{j2^{-n}} - Z_{(j-1)2^{-n}})^2 = f(y), \]

   where \(p\)-lim denotes a limit in probability of random variables.

   (b) Assume that \(f\) is piecewise differentiable on \([0, \infty)\) with piecewise constant derivative \(\sigma^2(s) := f'(s)\), say taking values \(\sigma^2_j\) on intervals \([y_j-1, y_j)\) for some \(0 = y_0 < y_1 < \ldots < y_n < \ldots\). Let \((W_y)_{y \geq 0}\) be a Brownian motion. Show that the process

   \[ \tilde{Z}_y = \int_0^y \sigma(r) dW_r := \sum_{i=1}^j (W_{y_i} - W_{y_{i-1}})\sigma_i + (W_y - W_{y_j})\sigma_{j+1}, \]

   \(y_j \leq y < y_{j+1}\), has the same distribution as \(Z\).

   \textit{This result holds in fact for a very wide class of stochastic processes \(\sigma\). This is why both time-change models and models where the Brownian motion coefficient \(\sigma\) varies with time are called stochastic volatility processes.}

2. Consider a Lévy process \((X_t)_{t \geq 0}\) with characteristics \((a, \sigma^2, g)\) and let \(f : [0, \infty) \to [0, \infty)\) be an increasing function. Set \(Z_y = X_{f(y)}, y \geq 0\).

   (a) Show that the \((Z_y)_{y \geq 0}\) has independent increments.

   (b) Show that \(y \mapsto Z_y\) is right-continuous with left limits if \(y \mapsto f(y)\) is right-continuous with left limits.

   (c) Show that \((Z_y)_{y \geq 0}\) has stationary increments if and only if either \(f\) is linear or \(X \equiv 0\).

   (d) Show that the distribution of \(Z_y\) is infinitely divisible. For each \(y \geq 0\), specify the characteristics in the Lévy-Khintchine representation of the distribution of \(Z_y\) as a random variable.

3. Let \((X_t)_{t \geq 0}\) be a compound Poisson process and \(f : [0, \infty) \to [0, \infty)\) right-continuous and increasing with \(f(0) = 0\) and \(f(\infty) = \infty\). Set \(Z_y = X_{f(y)}, y \geq 0\).

   (a) Suppose first that \(f\) is differentiable. Show that

   \[ N((a, b] \times (c, d]) = \# \{y \in (a, b] : \Delta Z_y \in (c, d]\}, \quad 0 \leq a < b, -\infty \leq c < d \leq \infty, \]

   is a Poisson counting measure and specify its intensity function \(g : [0, \infty) \times [0, \infty) \to [0, \infty)\). Deduce that for all \(y \geq 0\), we have \(P(\Delta Z_y = 0) = 1\). Explain how \((Z_y)_{y \geq 0}\) can be constructed from a Poisson point process with intensity function \(g\).
(b) If \( \Delta f(s) := f(s) - f(s-) > 0 \), calculate the moment generating function of \( \Delta Z_s \). What is \( \mathbb{P}(\Delta Z_s = 0) \)?

(c) If the function
\[
f_0(y) = f(y) - \sum_{0 \leq s \leq y} \Delta f(s), \quad y \geq 0
\]
is differentiable, show that \((Z_y)_{y\geq 0}\) has the same distribution as
\[
Z_y^0 + \sum_{0 \leq s \leq y} J_s, \quad y \geq 0,
\]
where \((Z_y^0)_{y\geq 0}\) is constructed from a Poisson point process as in (a), and the \(J_s, s \geq 0\), are independent with moment generating functions as in (b).

4. Let \((X_t)_{t\geq 0}\) be a Poisson process with jump times \((T_n)_{n\geq 1}\).

(a) Give examples of differentiable functions \(f_i\) as in A.6.1(b) for which \(X_{f_i(y)}\) does and does not have the same distribution as
\[
\int_0^y \sqrt{f'_i(s)} dX_s := \sum_{n=1}^{X_y} \sqrt{f'_i(T_n)}, \quad y \geq 0.
\]

(b) Find all such functions \(f_i\) in (a).

In fact, constant multiples of Brownian motion are the only Lévy processes for which the two distributions coincide for all such functions \(f\).

5. Let \((X_t)_{t\geq 0}\) be a Lévy process with probability density function \(f_t\) and \((\tau_y)_{y\geq 0}\) a subordinator with characteristics \((0, g_{\tau})\) (sum of jumps, no compensation!). Define
\[
g(z) = \int_0^\infty f_t(z) g_{\tau}(t) dt, \quad z \in \mathbb{R} \setminus \{0\}.
\]

(a) In the case \(\text{Var}(X_1) < \infty\) and \(\text{Var}(\tau_1) < \infty\), show that \(g\) satisfies the requirements of a Lévy density of a Lévy process.

(b) In the case where either \(\tau\) or \(X\) is compound Poisson, show that \(g\) also satisfies the requirements of a Lévy density of a Lévy process. More specifically, if \(X\) is a compound Poisson process with intensity \(\lambda\), then we have \(\mathbb{P}(X_t = 0) \geq e^{-\lambda t}\); assume that, in fact, \(\mathbb{P}(X_t = 0) = e^{-\lambda t}\) and that \(\mathbb{P}(X_t \in (a, b)) = \int_a^b f_t(x) dx\) for \((a, b) \neq 0\).