A.2 Poisson counting measures

Please hand in scripts by Monday 1 February 2010, 1.30pm, Department of Statistics. This problem sheet is based on material up to lecture 4 in the notes. The various parts of the problems can be attempted quite independently of one another, so if you are stuck on one part, please still attempt the others. M.Sc. students should attempt and hand in work at least for problems 2.(b)-(c) and 4.

1. (a) Let $(X_t)_{t\geq 0}$ be a Poisson process with rate $\lambda \in (0,\infty)$ and arrival times $T_1, T_2, \ldots$. Show that $N((c,d]) = \#\{n \in \mathbb{N} : c < T_n \leq d\}$ is a Poisson counting measure on $[0,\infty)$ with constant intensity λ.

   (b) Let N be a Poisson counting measure on $[0,\infty)$ with time-varying intensity $\lambda(t)$, $t \geq 0$, continuous. Denote $X_t = N([0,t])$ and $T_j = \inf\{t \geq 0 : X_t = j\}$, $j \geq 1$.

   (i) Show that $(X_t)_{t\geq 0}$ has independent increments.

   (ii) Show that $(X_t)_{t\geq 0}$ has stationary increments if and only if the intensity function $\lambda(t)$ is constant.

   (iii) Show that $(X_t)_{t\geq 0}$ has right-continuous paths with left limits.

   (iv) Calculate the distribution of $X_t - X_s$.

   (v) Calculate the survival function $\mathbb{P}(T_1 > s)$, $s \geq 0$, of T_1.

   (vi) Show that $T_2 - T_1$ is independent of T_1 if and only if the intensity function $\lambda(t)$ is constant. Calculate the joint density of $(T_1, T_2 - T_1)$.

2. (a) Show that for $\beta > 0$ and $\gamma < \beta$

$$\int_0^\infty (e^{\gamma x} - 1)\frac{1}{x}e^{-\beta x}dx = -\log\left(1 - \frac{\gamma}{\beta}\right)$$

   e.g. by suitable (and well-justified) differentiation under the integral sign.

   (b) Let $(\Delta_t)_{t\geq 0}$ be a Poisson point process with intensity function $\alpha x^{-1}e^{-\beta x}$. Use the exponential formula for Poisson point processes to show that

$$C_t = \sum_{s\leq t} \Delta_s \quad \text{has a Gamma distribution, density } \frac{\beta^{\alpha t}}{\Gamma(\alpha t)}x^{\alpha t-1}e^{-\beta x},\ x \geq 0.$$

   (c) Show that $(C_t)_{t\geq 0}$ as defined in (b) is a Lévy process.

3. Let X and Y be two independent increasing compound Poisson processes. Denote the respective jump rates by λ_X and λ_Y, assume that the jump size distributions are continuous with densities h_X and h_Y. Denote $D = X - Y$.

   (a) Show that X and Y have no jump times in common.

   (b) Show that D has jump times according to a Poisson process with rate $\lambda_X + \lambda_Y$.

   (c) Calculate the distribution of the first jump size of D.

4. Let X, Y and D be as in Exercise 3.

   (a) Show that $(\Delta X_t)_{t\geq 0}$ is a Poisson point process and specify its intensity function.

   (b) Show that $(\Delta D_t)_{t\geq 0}$ is a Poisson point process and specify its intensity function.

   (c) Deduce from (a) and (b) that D is also a compound Poisson process.

   (d) Show that *every* real-valued compound Poisson process C can be written uniquely as the difference of two *independent increasing* compound Poisson processes.

   Note that the theory of Poisson point processes applied here is neater than the conditioning in Exercise 3., which could also be developed and iterated to establish (c) *here. Intensity functions just add, jump size distributions are mixtures/weighted averages.*

Remark: Interchanging limits and expectation/integration/summation is not always permitted, and while we do not develop in this course the reasons why we may interchange, we add "by monotone convergence"*, whenever we have* increasing *or* decreasing *limits of* finite *quantities. General measure-theoretic statements have been established in B10a, special cases for Lebesgue integrals have been established in Part A Integration, which is also not a prerequisite for this course. As in BS3a, it is enough for our purposes to formulate special cases, whose statements do not require any of the formal technical setup. For convergence as $n \to \infty$ these are:*

- $Z_n \uparrow Z$ *and* $\mathbb{E}(|Z_n|) < \infty$ *for all* $n \in \mathbb{N}$ *implies*

$$\mathbb{E}(Z_n) \uparrow \mathbb{E}(Z) \in \mathbb{R} \cup \{\infty\}.$$

- $f_n \uparrow f$ *and* $\int_{\mathbb{R}} |f_n(x)|dx < \infty$ *for all* $n \in \mathbb{N}$ *implies*

$$\int_{\mathbb{R}} f_n(x)dx \uparrow \int_{\mathbb{R}} f(x)dx \in \mathbb{R} \cup \{\infty\}.$$

- $a_m^{(n)} \uparrow a_m$ *for all* $m \in \mathbb{N}$ *and* $\sum_{m=0}^{\infty} |a_m^{(n)}| < \infty$ *for all* $n \in \mathbb{N}$ *implies*

$$\sum_{m=0}^{\infty} a_m^{(n)} \uparrow \sum_{m=0}^{\infty} a_m \in \mathbb{R} \cup \{\infty\}.$$

- $\Delta_s^{(n)} \uparrow \Delta_s$ *and* $\sum_{s\leq t} |\Delta_s^{(n)}| < \infty$ *for all* $n \in \mathbb{N}$ *implies*

$$\sum_{s\leq t} \Delta_s^{(n)} \uparrow \sum_{s\leq t} \Delta_s \in \mathbb{R} \cup \{\infty\}.$$

The last statement is useful to show right-continuity and the existence of left limits of sums of Poisson point processes, such as in 2.(c)