Introduction to Lévy processes

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Matthias Winkel

Departmental lecturer

(Institute of Actuaries and Aon lecturer in Statistics)

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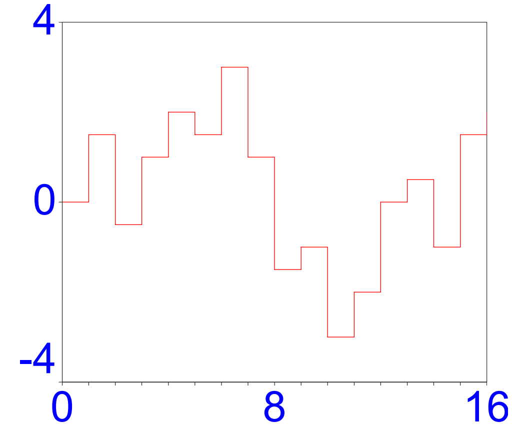
1. Random walks and continuous-time limits

Definition 1 Let Y_k, $k \geq 1$, be i.i.d. Then

$$S_n = \sum_{k=1}^{n} Y_k, \qquad n \in \mathbb{N},$$

is called a random walk.



Random walks have stationary and independent increments

$$Y_k = S_k - S_{k-1}, \qquad k \geq 1.$$

Stationarity means the Y_k have identical distribution.

Definition 2 A right-continuous process X_t, $t \in \mathbb{R}_+$, with stationary independent increments is called Lévy process.

What are S_n, $n \geq 0$, and X_t, $t \geq 0$?

Stochastic processes; mathematical objects, well-defined, with many nice properties that can be studied.
If you don't like this, think of a model for a stock price evolving with time. There are also many other applications.
If you worry about negative values, think of log's of prices.

What does Definition 2 mean?

Increments $X_{t_k} - X_{t_{k-1}}$, $k = 1, \ldots, n$, are independent and $X_{t_k} - X_{t_{k-1}} \sim X_{t_k - t_{k-1}}$, $k = 1, \ldots, n$ for all $0 = t_0 < \ldots < t_n$.
Right-continuity refers to the sample paths (realisations).

Can we obtain Lévy processes from random walks?

What happens e.g. if we let the time unit tend to zero, i.e. take a more and more remote look at our random walk?

If we focus at a fixed time, 1 say, and speed up the process so as to make n steps per time unit, we know what happens, the answer is given by the Central Limit Theorem:

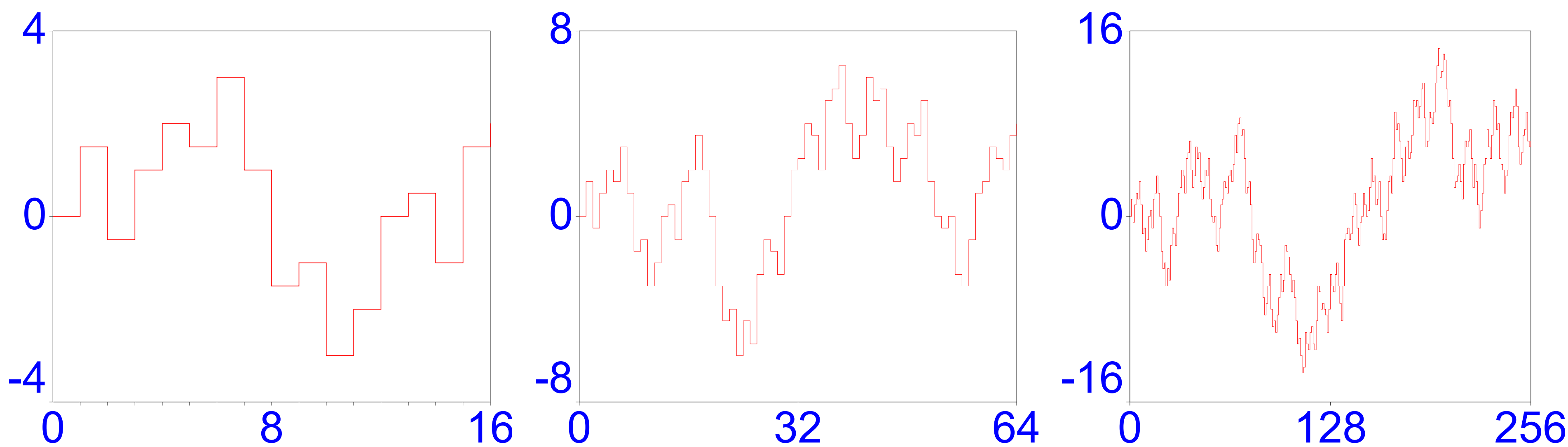
Theorem 1 (Lindeberg-Lévy) *If* $\sigma^2 = Var(Y_1) < \infty$, *then*

$$\frac{S_n - \mathbb{E}(S_n)}{\sqrt{n}} \to Z \sim N(0, \sigma^2) \qquad \textit{in distribution, as } n \to \infty.$$

Theorem 2 (Donsker) *If* $\sigma^2 = Var(Y_1) < \infty$, *then for* $t \geq 0$

$$X_t^{(n)} = \frac{S_{[nt]} - \mathbb{E}(S_{[nt]})}{\sqrt{n}} \to X_t \sim N(0, \sigma^2 t) \quad \textit{in distribution.}$$

Furthermore, $X^{(n)} \to X$ *where* X *is Brownian motion with diffusion coefficient* σ.



It can be shown that X has continuous sample paths.

We had to put the condition $Var(Y_1) < \infty$. If it fails:

Theorem 3 (Doeblin) *If* $\alpha = \sup\{\beta \geq 0 : \mathbb{E}(|Y_1|^\beta) < \infty\} \in (0,2]$ *and* $\mathbb{E}(Y_1) = 0$ *for* $\alpha > 1$, *then (under a weak regularity condition)*

$$\frac{S_n}{n^{1/\alpha}\ell(n)} \to Z \qquad \textit{in distribution, as } n \to \infty.$$

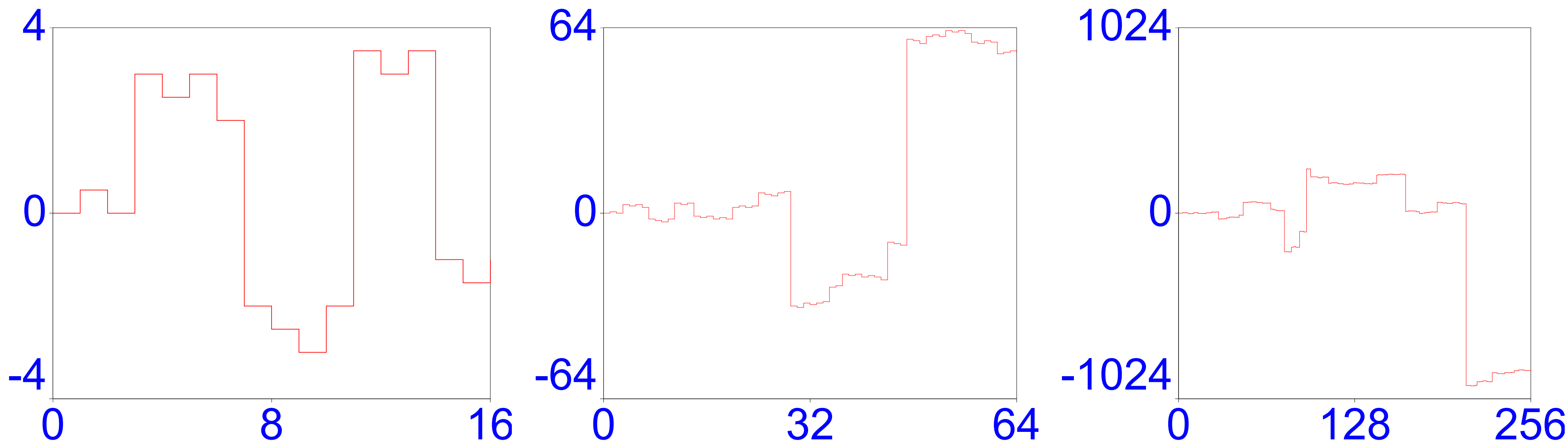
for a slowly varying ℓ. Z *has a so-called stable distribution.*

Think $\ell \equiv 1$. The family of stable distributions has three parameters, $\alpha \in (0,2]$, $c_\pm \in \mathbb{R}_+$, $\mathbb{E}(|Z|^\beta) < \infty \iff \beta < \alpha$ (or $\alpha = 2$).

Theorem 4 *If* $\alpha = \sup\{\beta \geq 0 : \mathbb{E}(|Y_1|^\beta) < \infty\} \in (0,2]$ *and* $\mathbb{E}(Y_1) = 0$ *for* $\alpha > 1$, *then (under a weak regularity cond.)*

$$X_t^{(n)} = \frac{S_{[nt]}}{n^{1/\alpha}\ell(n)} \to X_t \qquad \textit{in distribution, as } n \to \infty.$$

Also, $X^{(n)} \to X$ *where* X *is a so-called stable Lévy process.*



One can show that $(X_{ct})_{t\geq 0} \sim (c^{1/\alpha}X_t)_{t\geq 0}$ (scaling).

To get to full generality, we need triangular arrays.

Theorem 5 (Khintchine) *Let $Y_k^{(n)}$, $k = 1, \ldots, n$, be i.i.d. with distribution changing with $n \geq 1$, and such that*

$$Y_1^{(n)} \to 0, \qquad \text{in probability, as } n \to \infty.$$

If

$$S_n^{(n)} \to Z \qquad \text{in distribution,}$$

then Z has a so-called infinitely divisible distribution.

Theorem 6 (Skorohod) *In Theorem 5, $k \geq 1$, $n \geq 1$,*

$$X_t^{(n)} = S_{[nt]}^{(n)} \to X_t \qquad \text{in distribution, as } n \to \infty.$$

Furthermore, $X^{(n)} \to X$ where X is a Lévy process.

Do we really want to study Lévy processes as limits?

No! Not usually. But a few observations can be made:

Brownian motion is a very important Lévy process.

Jumps seem to be arising naturally (in the stable case).

We seem to be restricting the increment distributions to:

Definition 3 A r.v. Z has an infinitely divisible distribution if $Z = \tilde{S}_n^{(n)}$ for all $n \geq 1$ and suitable random walks $\tilde{S}^{(n)}$.

Example for Theorems 5 and 6: $B(n, p_n) \to Poi(\lambda)$ for $np_n \to \lambda$ is a special case of Theorem 5 ($Y_k^{(n)} \sim B(1, p_n)$), and Theorem 6 turns out to give an approximation of the Poisson process by Bernoulli random walks.

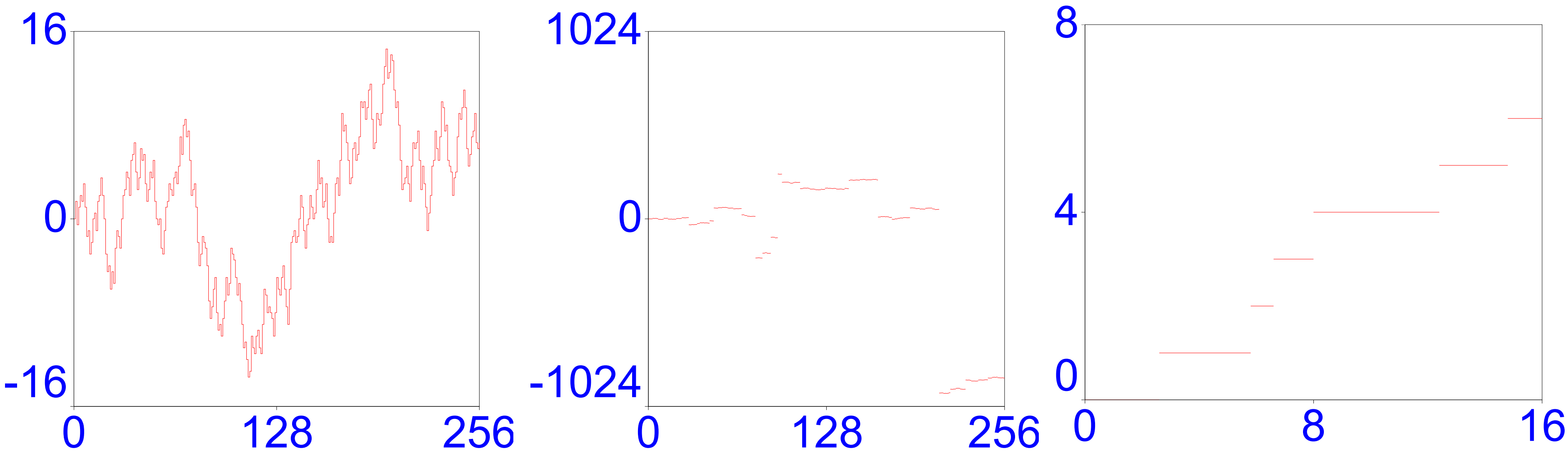
2. Examples

Example 1 (Brownian motion) $X_t \sim N(0, t)$.

Example 2 (Stable process) X_t stable.

Example 3 (Poisson process) $X_t \sim Poi(t/2)$

To emphasise the presence of jumps, we remove the vertical lines.



3. Classification and construction of Lévy proc.

Why restrict to infinitely divisible increment distrib.?

You have restrictions for random walks: $n-$step increments S_n must be divisible into n iid increments, for all $n \geq 2$.

For a Lévy process, X_t, $t > 0$, must be divisible into $n \geq 2$ iid random variables

$$X_t = \sum_{k=1}^{n} (X_{tk/n} - X_{t(k-1)/n}),$$

since these are successive increments (hence independent) of equal length (hence identically distrib., by stationarity).

Approximations are cumbersome. Often nicer direct arguments exist in continuous time, e.g., because the class of infinitely divisible distributions can be parametrized nicely.

Theorem 7 (Lévy-Khintchine) *A r.v. Z is infinitely divisible iff its characteristic function $\mathbb{E}(e^{i\lambda Z})$ is of the form*

$$\exp\left\{i\beta\lambda - \frac{1}{2}\sigma^2\lambda^2 + \int_{\mathbb{R}^*}\left(e^{i\lambda x} - 1 - i\lambda x 1_{\{|x|\le 1\}}\right)\nu(dx)\right\}$$

where $\beta \in \mathbb{R}$, $\sigma^2 \ge 0$ and ν is a measure on $\mathbb{R}^*$ such that

$$\int_{\mathbb{R}^*}(1 \wedge x^2)\nu(dx) < \infty, \qquad [\nu(dx) \approx f(x)dx.]$$

'Only if' is hard to prove. We give an indication of 'if' by constructing the associated Lévy processes.

The hidden beauty of this abstract parametrization can only be appreciated when interpreted for the Lévy process with given infinitely divisible increment distribution.

Theorem 8 (Lévy-Itô) *Let X be a Lévy process, the distribution of X_1 parametrized by (β, σ^2, ν). Then X decomposes*

$$X_t = \beta t + \sigma B_t + J_t + M_t$$

where B is a Brownian motion, and $\Delta X_t = X_t - X_{t-}$, $t \geq 0$, an independent Poisson point process with intensity measure ν,

$$J_t = \sum_{s \leq t} \Delta X_s 1_{\{|\Delta X_s| > 1\}}$$

and M is a martingale with jumps $\Delta M_t = \Delta X_t 1_{\{|\Delta X_s| \leq 1\}}$.

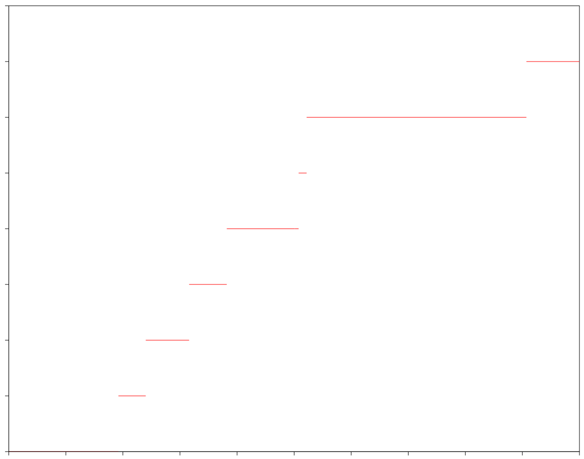
4. Examples

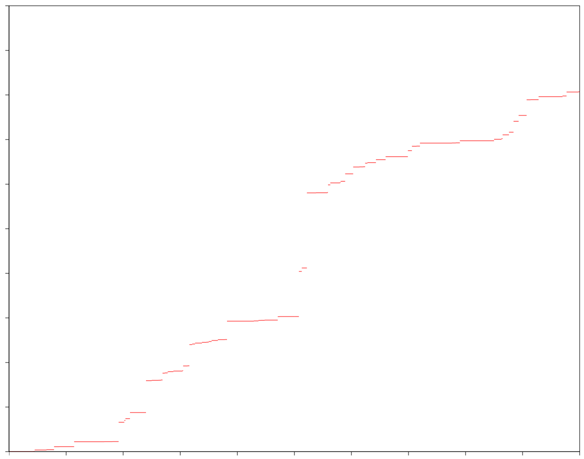
$$Z \geq 0 \iff E(e^{-\lambda Z}) = \exp\left\{-\beta'\lambda - \int_{(0,\infty)} \left(1 - e^{-\lambda x}\right) \nu(dx)\right\}$$

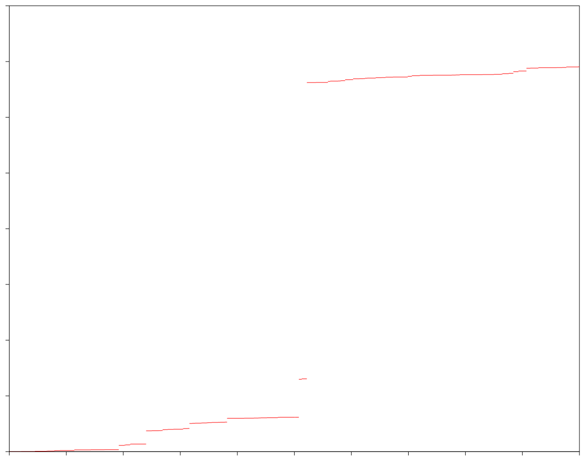
Example 4 (Poisson proc.) $\beta' = 0$, $\nu(dx) = \lambda\delta_1(dx)$.

Example 5 (Gamma process) $\beta' = 0$, $\nu(x) = f(x)dx$ with $f(x) = ax^{-1}\exp\{-bx\}$, $x > 0$. Then $X_t \sim Gamma(at, b)$.

Example 6 (stable subordinator) $\beta' = 0$, $f(x) = cx^{-3/2}$



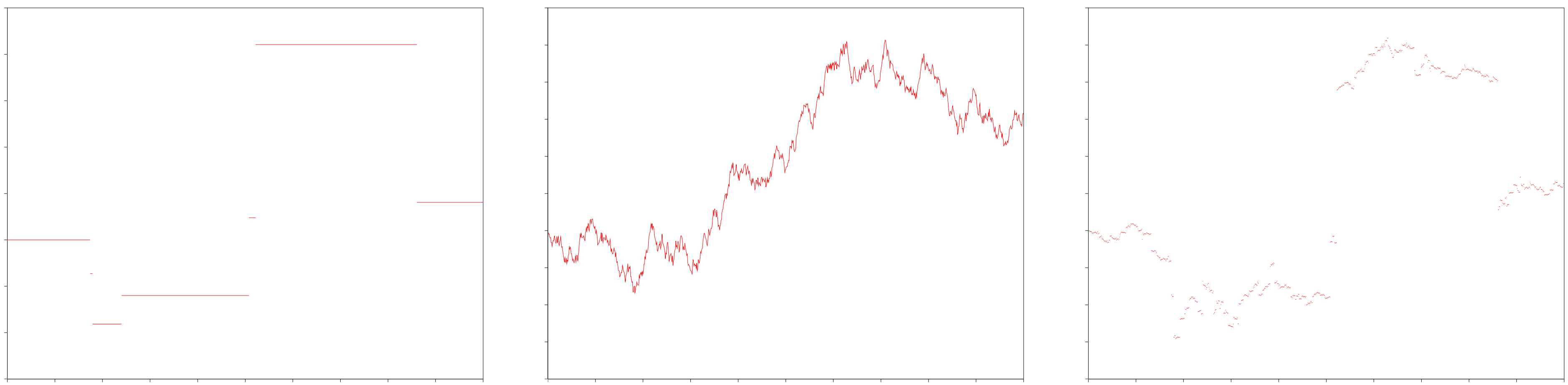


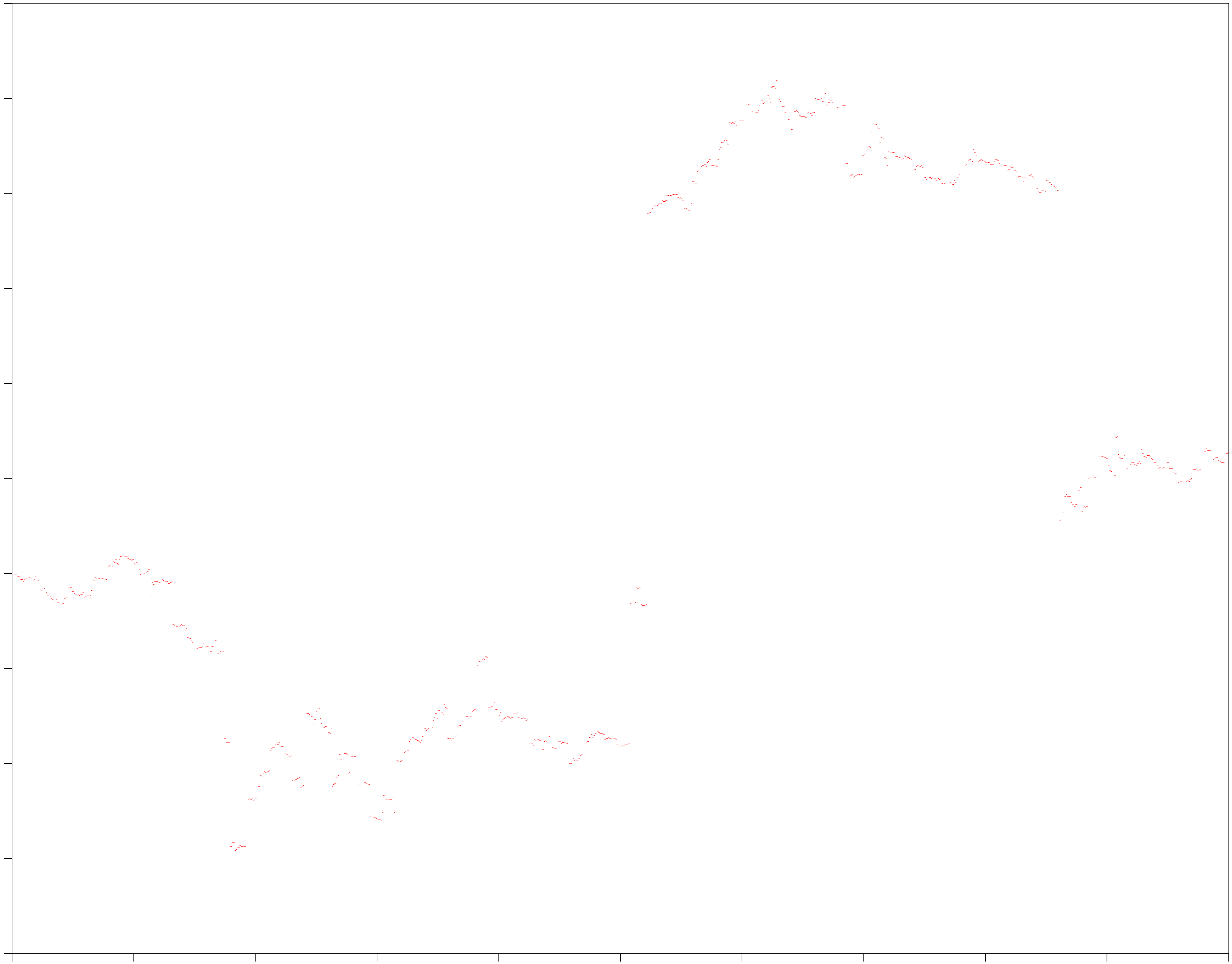


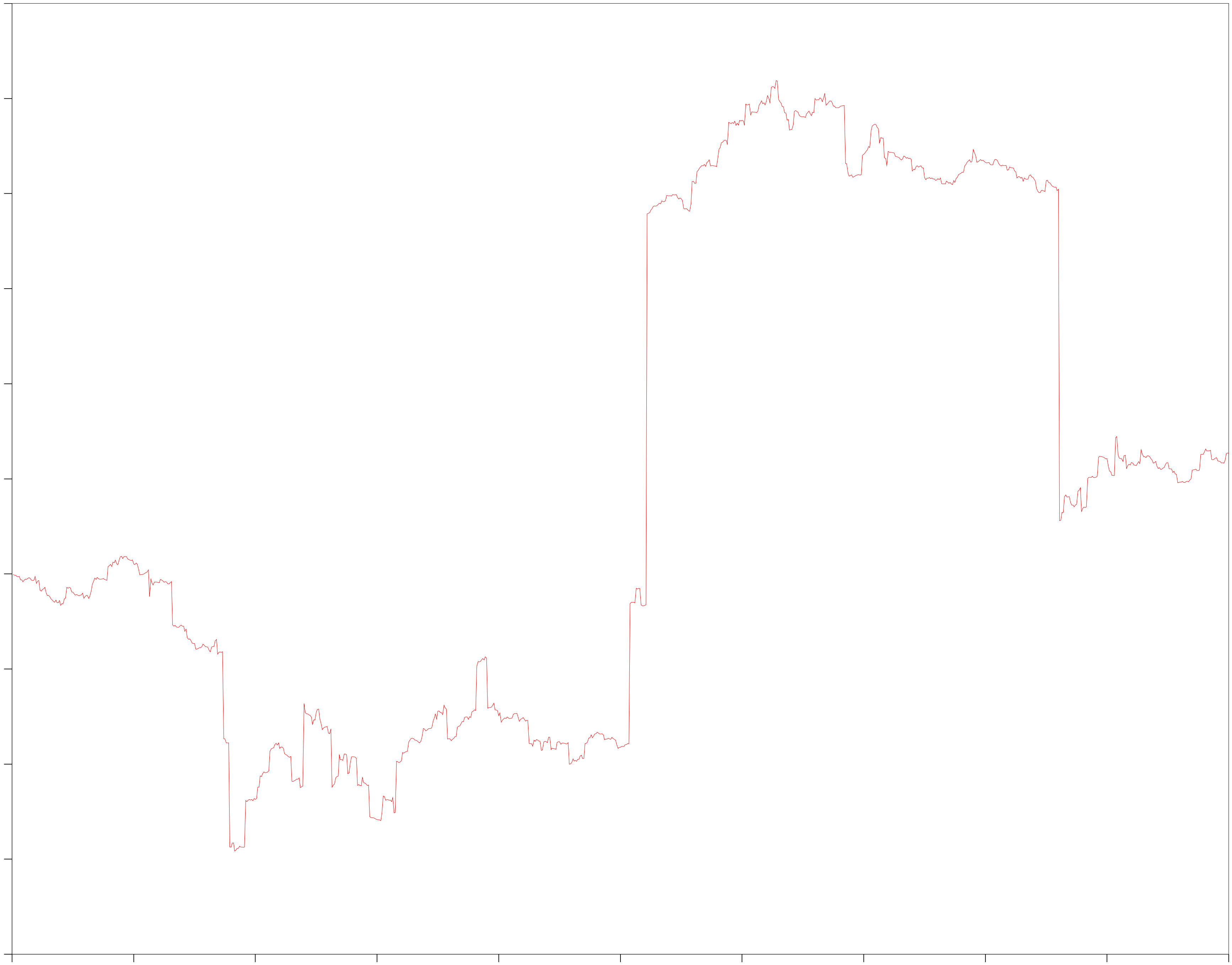
Example 7 (Compound Poisson process) $\beta = \sigma^2 = 0$. Choose jump distribution, e.g. density $g(x)$, intensity $\lambda > 0$, $\nu(dx) = \lambda g(x)dx$. J in Theorem 8 is compound Poisson.

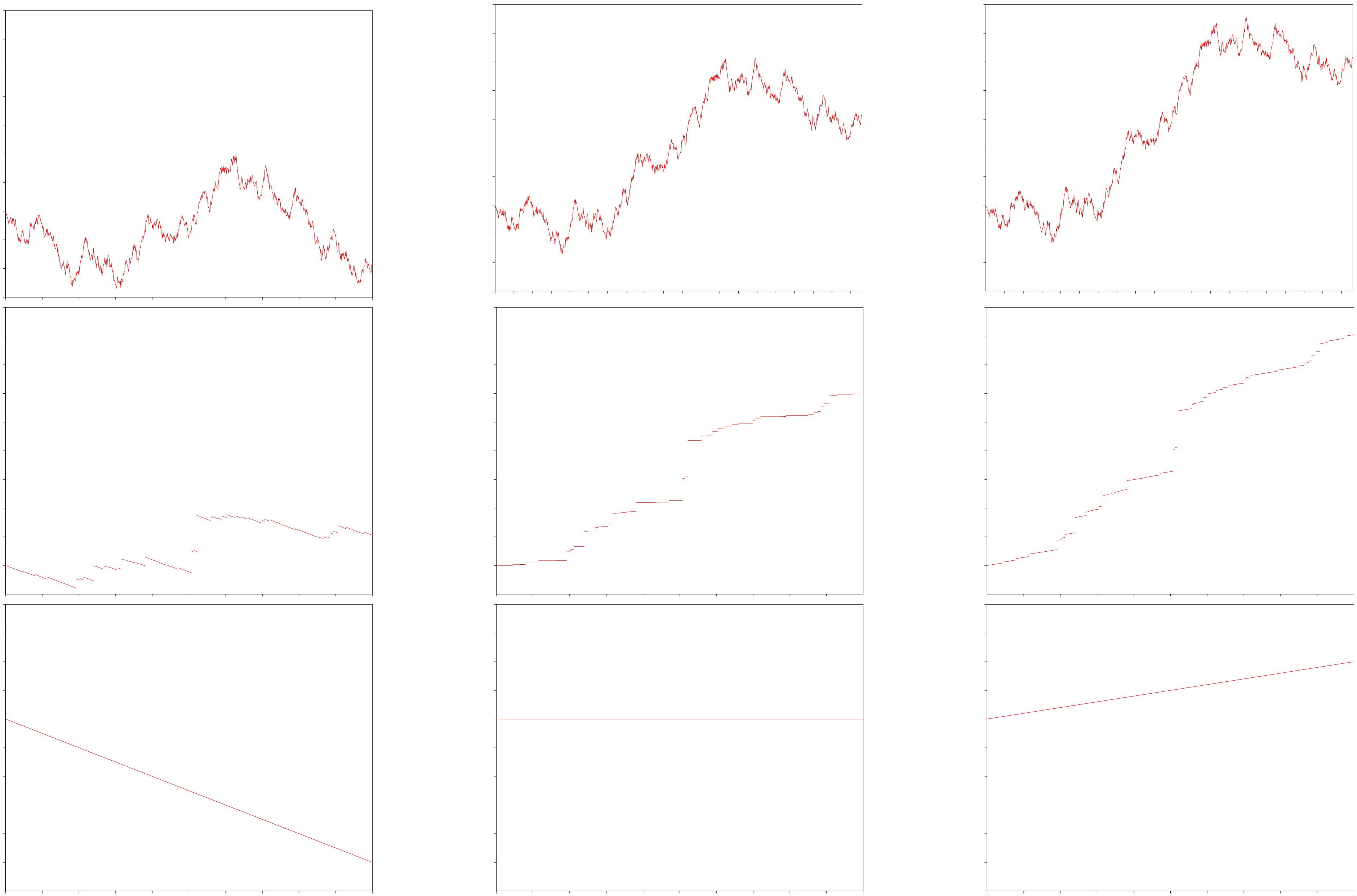
Example 8 (Brownian motion) $\beta = 0$, $\sigma^2 > 0$, $\nu \equiv 0$.

Example 9 (Cauchy process) $\beta = \sigma^2 = 0$, $\nu(dx) = f(x)dx$ with $f(x) = x^{-2}$, $x > 0$, $f(x) = |x|^{-2}$, $x < 0$.









$\beta = -5$ $\beta = 0$ $\beta = 2$

5. Poisson point processes and simulation

What does it mean that $(\Delta X_t)_{t\geq 0}$ is a Poisson point process with intensity measure ν?

Definition 4 A stochastic process $(H_t)_{t\geq 0}$ in a measurable space $\mathcal{E} = \mathcal{E}^* \cup \{0\}$ is called a Poisson point process with intensity measure ν on $\mathcal{E}^*$ if

$$N_t(A) = \#\{s \leq t : H_s \in A\}, \quad t \geq 0, A \subset \mathcal{E}^* \text{ measurable}$$

satisfies

- $N_t(A)$, $t \geq 0$, is a Poisson process with intensity $\nu(A)$.
- For $A_1, \ldots, A_n$ disjoint, the processes $N_t(A_1), \ldots, N_t(A_n)$, $t \geq 0$, are independent.

$N_t(A)$, $t \geq 0$, counts the number of points in A, but does not tell where in A they are. Their distribution on A is ν:

Theorem 9 (Itô) *For all measurable $A \subset \mathcal{E}^*$ with $M = \nu(A) < \infty$, denote the jump times of $N_t(A)$, $t \geq 0$, by*

$$T_n(A) = \inf\{t \geq 0 : N_t(A) = n\}, \qquad n \geq 1.$$

Then

$$Z_n(A) = H_{T_n(A)}, \qquad n \geq 1,$$

are independent of $N_t(A)$, $t \geq 0$, and iid with common distribution $M^{-1}\nu(\cdot \cap A)$.

This is useful to simulate H_t, $t \geq 0$.

Simulation of $(\Delta X_t)_{t\geq 0}$

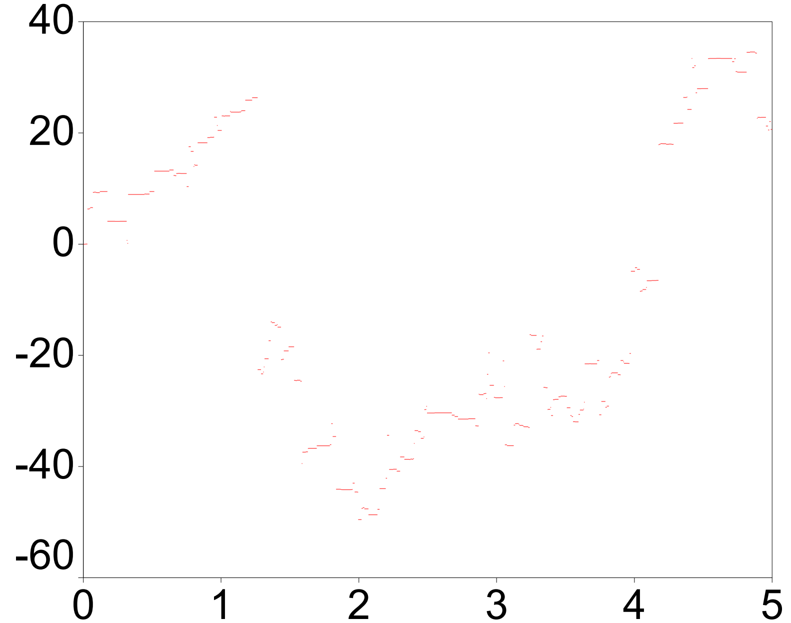
Time interval $[0,5]$

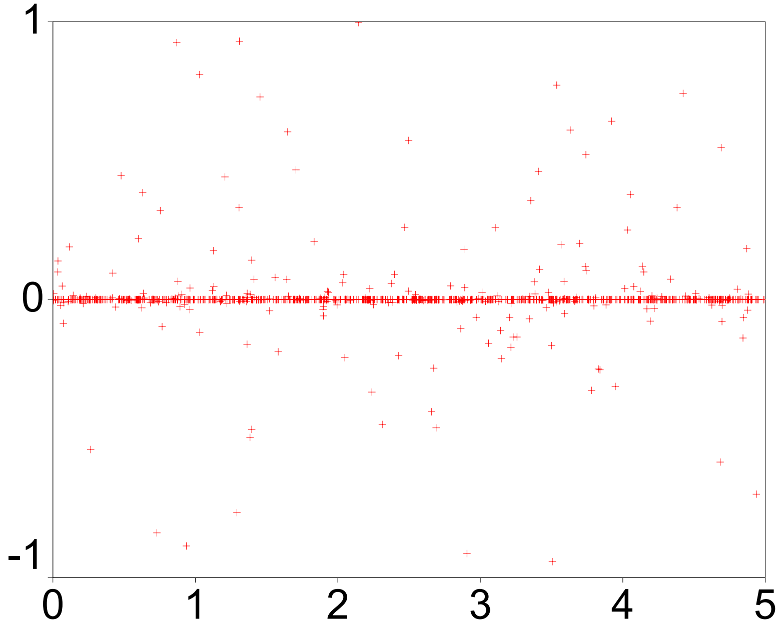
$\varepsilon > 0$ small, $M = \nu((-\varepsilon,\varepsilon)^c) < \infty$

$N \sim Poi(5\nu((-\varepsilon,\varepsilon)^c))$

$\tilde{T}_1, \ldots, \tilde{T}_N \sim U(0,5)$

$\Delta X_{\tilde{T}_j} \sim M^{-1}\nu(\cdot \cap (-\varepsilon,\varepsilon)^c)$





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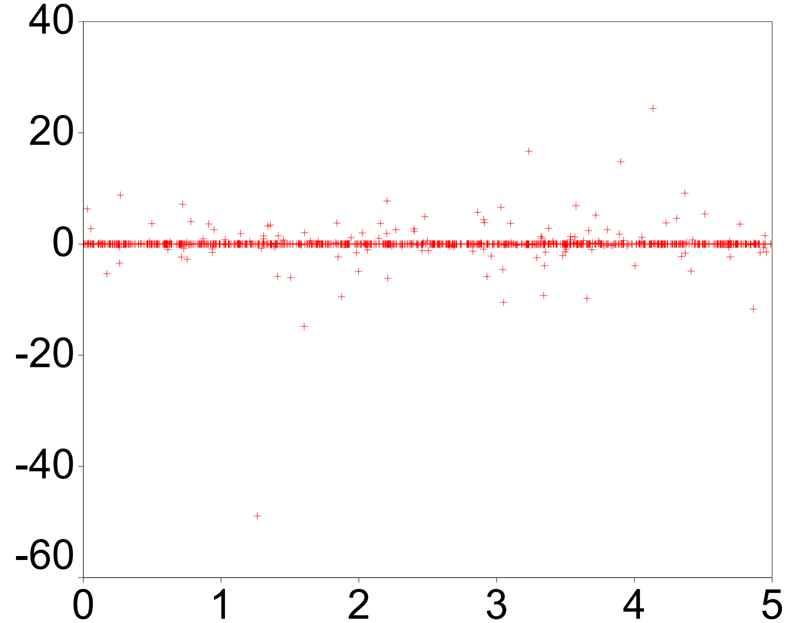


Corrections to improve approximation

First, add independent Brownian motion $\beta t + \sigma B_t$.

If X symmetric, small ε means small error.

If X not symmetric, we need a drift correction $-\gamma t$

$$\gamma = \int_{A \cap [-1,1]} x \nu(dx), \qquad \text{where } A = (-\varepsilon, \varepsilon)^c.$$

Also, one can often add a Normal variance correction τC_t

$$\tau^2 = \int_{(-\varepsilon,\varepsilon)} x^2 \nu(dx), \quad C \text{ independent Brownian motion}$$

to account for the many small jumps thrown away. This can be justified by a version of Donsker's theorem and

$$\mathbb{E}(X_1) = \beta + \int_{[-1,1]^c} x\nu(dx), \quad Var(X_1) = \sigma^2 + \int_{\mathbb{R}^*} x^2 \nu(dx).$$

Decomposing a general Lévy process X

Start with intervals of jump sizes in sets A_n with

$$\bigcup_{n\geq 1} A_n = (0,\infty), \quad A_{-n} = -A_n, \quad \bigcup_{n\leq -1} A_n = (-\infty, 0)$$

so that $\nu(A_n + A_{-n}) \geq 1$, say. We construct $X^{(n)}$, $n \in \mathbb{Z}^*$, independent Lévy processes with jumps in A_n according to $\nu(\cdot \cap A_n)$ (and drift correction $-\gamma_n t$). Now

$$X_t = \sum_{n\in\mathbb{Z}} X_t^{(n)}, \qquad \text{where } X_t^{(0)} = \beta t + \sigma B_t.$$

In practice, you may cut off the series when $X_t^{(n)}$ is small, and possibly estimate the remainder by a Brownian motion.

A spectrally negative Lévy process: $\nu((0,\infty)) = 0$

$\sigma^2 = 0$,

$\nu(dx) = f(x)dx$,

$f(x) = |x|^{-5/2}$,

$x \in [-3, 0)$,

$\beta = 0.845$

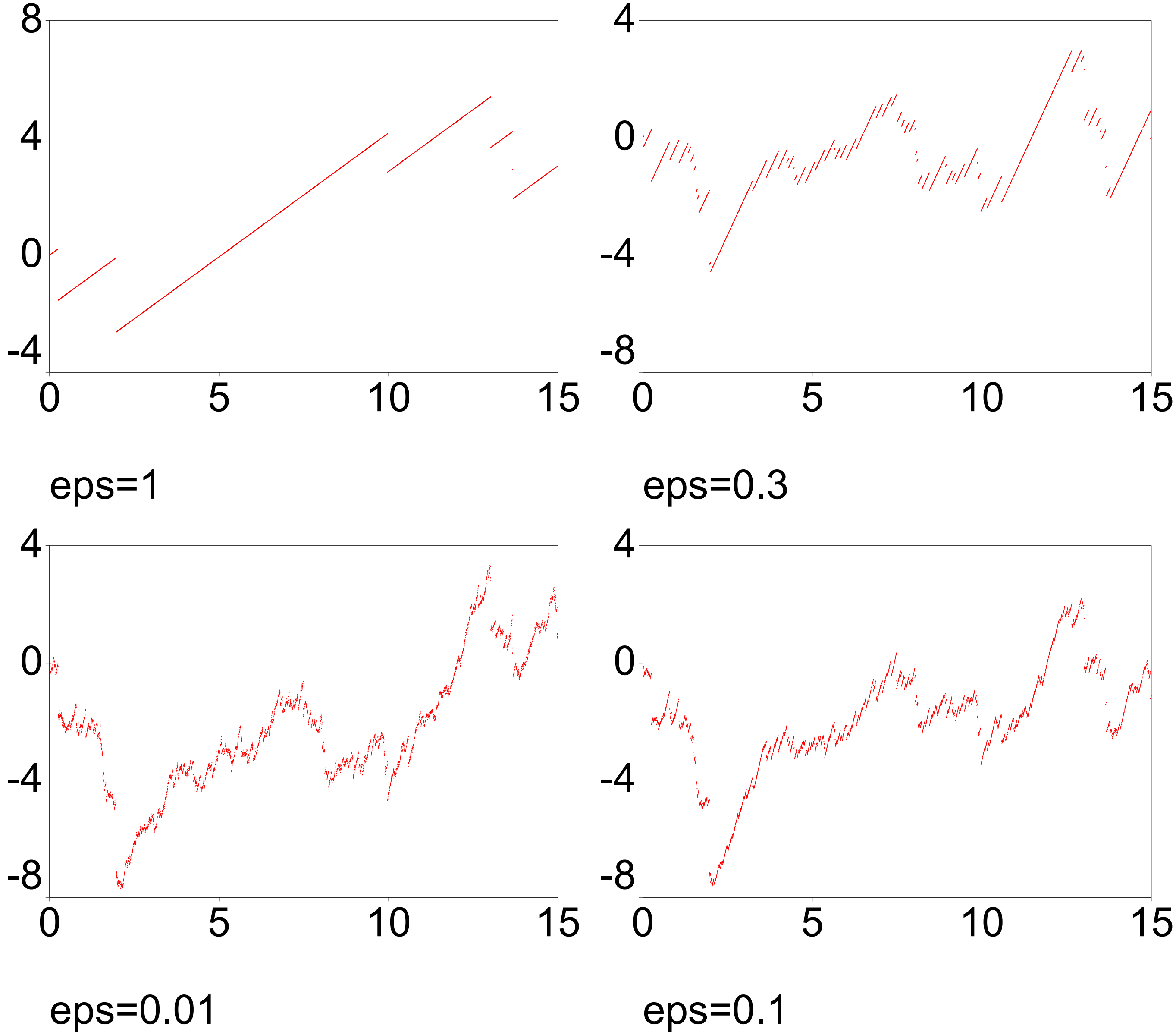
$\gamma_1 = 0$,

$\gamma_{0.3} = 1.651$,

$\gamma_{0.1} = 4.325$,

$\gamma_{0.01} = 18$,

$\gamma_\varepsilon \uparrow \infty$ as $\varepsilon \downarrow 0$.



Another look at the Lévy-Khintchine formula

$$\mathbb{E}(e^{i\lambda X_1}) = \mathbb{E}\left(\exp\left\{i\lambda \sum_{n\in\mathbb{Z}} X_1^{(n)}\right\}\right) = \prod_{n\in\mathbb{Z}} \mathbb{E}(e^{i\lambda X_1^{(n)}})$$

can be calculated explicitly, for $n \in \mathbb{Z}^*$ ($n = 0$ obvious)

$$\mathbb{E}(e^{i\lambda X_1^{(n)}}) = \mathbb{E}\left(\exp\left\{i\lambda\left(-\gamma_n + \sum_{k=1}^{N} Z_k\right)\right\}\right)$$

$$= \exp\left\{-i\lambda \int_{A_n\cap[-1,1]} x\nu(dx)\right\} \sum_{m=0}^{\infty} \mathbb{P}(N=m)\left(\mathbb{E}(e^{i\lambda Z_1})\right)^m$$

$$= \exp\left\{\int_{A_n}\left(e^{i\lambda x} - 1 - i\lambda x 1_{\{|x|\le 1\}}\right)\nu(dx)\right\}$$

to give for $\mathbb{E}(e^{i\lambda X_1})$ the Lévy-Khintchine formula

$$\exp\left\{i\beta\lambda - \frac{1}{2}\sigma^2\lambda^2 + \int_{\mathbb{R}^*}\left(e^{i\lambda x} - 1 - i\lambda x 1_{\{|x|\le 1\}}\right)\nu(dx)\right\}$$

Summary

We've defined Lévy processes via stationary independent increments.

We've seen how Brownian motion, stable processes and Poisson processes arise as limits of random walks, indicated more general results.

We've analysed the structure of general Lévy processes and given representations in terms of compound Poisson processes and Brownian motion with drift.

We've simulated Lévy processes from their marginal distributions and from their Lévy measure.