A.3 Arbitrage-free pricing

1. There are two securities S^1 and S^2 in a market. The values of both securities at $t = 0$ is 1. The value of S^1 at $t = 1$ is $1 + i$. The value of S^2 at $t = 1$ is either u or d, each with positive probability. Assume $0 < d < u$.

   Show that an arbitrage opportunity exists in this model if and only if either $1+i \geq u$ or $1 + i \leq d$.

2. Consider a one-period securities market model consisting of two assets. At the end of the period, the economy will be either "up" or "down". The first asset pays £5 if the economy is up and £1.25 if the economy is down. The second asset pays £1.25 if the economy is up and £5 if the economy is down. At $t = 0$, the first asset sells for £2.75 and the second for £1.90. An investor would like to receive £1.25 if the economy is up and nothing if the economy is down.

   (a) What are the portfolio holdings of the two assets that have the payoff the investor would like, if any?

   (b) If there is one, what should the price of the portfolio be?

3. (a) Using the no arbitrage assumption, show from first principles that the forward price F at $t = 0$ for an asset S, with no income, with value S_0 at $t = 0$, where there is a risk-free asset with constant force of interest, r, with maturity at time $T > 0$, is given by $F = S_0 e^{rT}$. Assume that there are no transaction costs.

   (b) Extend the argument in (a) to derive the forward price where the asset provides a fixed known (cash flow of) income.

4. The forward rate from time t to time $t + 1$, $f_{t,1}$, has the following values

$$f_{0,1} = 4.0\%, \quad f_{1,1} = 4.5\%, \quad f_{2,1} = 4.8\%.$$

   (a) Assuming no arbitrage, calculate

      i. the price per £100 nominal of a 3-year bond paying an annual coupon in arrears of 5%, redeemed at par in exactly three years, and

      ii. the gross redemption yield from the bond.

   (b) Explain why a bond with a higher coupon would have a lower gross redemption yield, for the same term to redemption.

5. An asset has a current price of 100p. It will pay an income of 5p in 20 days' time. Given a risk-free rate of interest of 6% per annum convertible half-yearly and assuming no arbitrage, calculate the forward price to be paid in 40 days.

6. $f_{t,r}$ is the forward rate applicable over the period t to $t + r$. i_t is the spot rate over the period 0 to t. The gross redemption yield from a one-year bond with a 6% annual coupon is 6% per annum effective; the gross redemption yield from a two-year bond with a 6% annual coupon is 6.3%; and the gross redemption yield

from a three year bond with a 6% annual coupon is 6.6% per annum effective. All the bonds are redeemed at par and are exactly one year from the next coupon payment.

(a) i. Calculate i_1, i_2 and i_3 assuming no arbitrage.
ii. Calculate $f_{0,1}$, $f_{1,1}$ and $f_{2,1}$ assuming no arbitrage.

(b) Explain why the forward rates increase more rapidly with term than the spot rates.