## A.1 Revision, lifetime distributions

*Please hand in your work by Tuesday 31 January 2012 4pm, at the Department of Statistics.*

1. (a) Let $L_1, \ldots, L_n$ be independent $\mathrm{Exp}(\lambda)$ random variables. Show that the maximum likelihood estimator for $\lambda$ is given by

$$\hat{\lambda} = \frac{n}{L_1 + \ldots + L_n}.$$

(b) The following data resulted from a life test of refrigerator motors (hours to burnout):

| **Hours to burnout** | | | | |
|---|---|---|---|---|
| 104.3 | 158.7 | 193.7 | 201.3 | 206.2 |
| 227.8 | 249.1 | 307.8 | 311.5 | 329.6 |
| 358.5 | 364.3 | 370.4 | 380.5 | 394.6 |
| 426.2 | 434.1 | 552.6 | 594.0 | 691.5 |

i. Assuming refrigerator motors have $\mathrm{Exp}(\lambda)$ lifetimes, give the maximum likelihood estimate for $\lambda$.

ii. Still assuming $\mathrm{Exp}(\lambda)$ lifetimes, calculate the Fisher information and construct approximate 95% confidence intervals for $\lambda$ and $1/\lambda$ using the approximate Normal distribution of the maximum likelihood estimator.

iii. Still assuming $\mathrm{Exp}(\lambda)$ lifetimes, show that $2n\lambda/\hat{\lambda} \sim \chi^2_{2n}$. Let $a$ be such that $\mathbb{P}(2n\lambda/\hat{\lambda} \le a) = \alpha/2$ and $b$ such that $\mathbb{P}(2n\lambda/\hat{\lambda} \ge b) = \alpha/2$. Deduce an exact 95% confidence interval for $1/\lambda$.

iv. Produce a histogram of the data and comment.

v. Merge columns of your histogram appropriately to test whether the hypothesis of $\mathrm{Exp}(\lambda)$ lifetimes can be rejected. Use a $\chi^2$ goodness of fit test.

2. (a) Let $T_1, \ldots, T_m$ be independent continuous nonnegative random variables with hazard functions $h_1(\cdot), \ldots, h_m(\cdot)$. Prove that $T = \min(T_1, \ldots, T_m)$ has hazard function $h_1(\cdot) + \ldots + h_m(\cdot)$.

(b) Let $T_1, \ldots T_m$ be independent random variables with Weibull distributions with rate parameters $k_1, \ldots k_m$ and common exponent $n$. Prove that $T = \min(T_1, \ldots, T_m)$ also has a Weibull distribution with exponent $n$.

(c) Calculate the hazard function of the truncated exponential distribution with maximal age $\omega$, and calculate the limit (in distribution) as $\lambda \downarrow 0$.

3. (a) Show that for a random variable $T$ which has as its distribution a mixture of exponential distributions $f_{T|M=\lambda}(t) = \lambda e^{-\lambda t}$, with mixing variable (random parameter) $M$, the unconditional mean and variance are given by

$$\mathbb{E}(T) = \mathbb{E}\left(\frac{1}{M}\right) \qquad \text{and} \qquad \mathrm{Var}(T) = 2\mathbb{E}\left(\frac{1}{M^2}\right) - \left(\mathbb{E}\left(\frac{1}{M}\right)\right)^2,$$

and the (unconditional) survival function of $T$ is given by

$$\bar{F}_T(t) = \mathcal{M}_M(-t), \qquad \text{where } \mathcal{M}_M(c) = \mathbb{E}(e^{cM})$$

is the moment generating function of $M$.

(b) Now take as mixing distribution a Gamma distribution with parameters $\alpha$ and $\nu$, i.e. $f_M(\lambda) = \nu^\alpha \lambda^{\alpha-1} e^{-\nu\lambda}/\Gamma(\alpha)$. Show that the corresponding mixture of exponential distributions has density

$$f_T(t) = \frac{\alpha\nu^\alpha}{(t+\nu)^{\alpha+1}}$$

Also calculate the survival function and hazard rate.

(c) Show that the Gompertz-Makeham distribution with hazard function

$$h(t) = \rho_0 + \rho_1 e^{\rho_2 t}$$

can be obtained as an exponential mixture provided $\rho_2 \leq 0$, and determine the distribution of the mixing random variable $M$. Hint: Calculate the moment generating function of a Poi($\nu$) random variable $\tilde{M}$ and adjust as necessary.

4. Suppose we observe $\ell_0$ independent and identically distributed lifetimes and consider the random variables behind associated lifetable entries $d_x$ and $\ell_x$, $x \geq 0$.

(a) Show that $\mathbb{E}(d_x - q_x\ell_x) = 0$ and $\mathrm{Var}(d_x - q_x\ell_x) = q_x(1-q_x)\mathbb{E}(\ell_x)$. Hint: Condition on $\ell_x$. What is the conditional distribution of $d_x$ given $\ell_x$?

(b) Is $\hat{q}_0^{(0)} = d_0/\ell_0$ unbiased? What about $\hat{q}_1^{(0)} = d_1/\ell_1$? Calculate the (approximate) Fisher Information matrix, the (approximate) variances of $\hat{q}_0^{(0)}$ and $\hat{q}_1^{(0)}$, and their estimates induced by the maximum likelihood estimates of $\hat{q}_x^{(0)}$.