Lecture 15

Markov models for insurance

Reading: Ross 7.10; CT4 Unit 6
Further reading: Norris 5.3

15.1 The insurance ruin model

Insurance companies deal with large numbers of insurance policies at risk. They are grouped according to type and various other factors into so-called portfolios. Let us focus on such a portfolio and model the associated claim processes, the claim sizes and the reserve process. We make the following assumptions.

- Claims arrive according to a Poisson process \((X_t)_{t \geq 0}\) with rate \(\lambda\).
- Claim amounts \((A_j)_{j \geq 1}\) are positive, independent of the arrival process and identically distributed with common probability density function \(k(a), a > 0\), and mean \(\mu = E(A_1)\).
- The insurance company provides an initial reserve of \(u \geq 0\) money units.
- Premiums are paid continuously at constant rate \(c\) generating a linear premium income accumulating to \(ct\) at time \(t\). We assume \(c > \lambda \mu\) to have more premium income than claim outgo, on average.
- We ignore all expenses and other influences.

In this setting, we define the following objects of interest

- The aggregate claims process \(C_t = \sum_{n=1}^{X_t} A_n, t \geq 0\).
- The reserve process \(R_t = u + ct - C_t, t \geq 0\).
- The ruin probability \(\psi(u) = P_u(R_t < 0 \text{ for some } t \geq 0)\), as a function of \(R_0 = u \geq 0\).
15.2 Aggregate claims and reserve processes

Proposition 110 \( C \) and \( R \) have stationary independent increments. Their moment generating functions are given by

\[
\mathbb{E}(e^{\gamma C_t}) = \exp \left\{ \lambda t \int_0^\infty (e^{\gamma a} - 1) k(a) da \right\}
\]

and

\[
\mathbb{E}(e^{\beta R_t}) = \exp \left\{ \beta u + \beta ct - \lambda t \int_0^\infty (1 - e^{-\beta a}) k(a) da \right\}.
\]

Proof: First calculate the moment generating function of \( C_t \):

\[
\mathbb{E}(e^{\gamma C_t}) = \mathbb{E} \left( \exp \left\{ \gamma \sum_{j=1}^{X_t} A_j \right\} \right)
= \sum_{n \in \mathbb{N}} \mathbb{E} \left( \exp \left\{ \gamma \sum_{j=1}^{n} A_j \right\} \right) \mathbb{P}(X_t = n)
= \sum_{n \in \mathbb{N}} \left( \mathbb{E}(e^{\gamma A_1}) \right)^n \mathbb{P}(X_t = n)
= \exp \left\{ \lambda t \left( \mathbb{E}(e^{\gamma A_1}) - 1 \right) \right\}
\]

which in the case where \( A_1 \) has a density \( k \), gives the formula required. The same calculation for the joint moment generating function of \( C_t \) and \( C_{t+s} - C_t \), or more increments, yields stationarity and independence of increments (only using the stationarity and independence of increments of \( X \), and the independence of the \((A_j)_{j \geq 1}\)).

The statements for \( R \) follow easily with \( \beta = -\gamma \). \( \square \)

The moment generating function is useful to calculate moments.

Example 111 We differentiate the moment generating functions at zero to obtain

\[
\mathbb{E}(C_t) = \frac{\partial}{\partial \gamma} \mathbb{E}(e^{\gamma A_1} - 1) \bigg|_{\gamma=0} = \lambda t \mathbb{E}(e^{\gamma A_1}) \bigg|_{\gamma=0} = \lambda t \mu.
\]

and \( \mathbb{E}(R_t) = u + ct - \lambda t \mu = u + (c - \lambda \mu)t \). Note that the Strong Law of Large Numbers, applied to increments \( Z_n = R_n - R_{n-1} \) yields

\[
\frac{R_n}{n} = \frac{u}{n} + \frac{1}{n} \sum_{j=1}^{n} Z_j \to \mathbb{E}(Z_1) = c - \lambda \mu > 0 \quad \text{a.s., as } n \to \infty.
\]

confirming our claim that \( c > \lambda \mu \) means that, on average, there is more premium income than claim outgo. In particular, this implies \( R_n \to \infty \) a.s. as \( n \to \infty \). Does this imply that \( \inf \{ R_s : 0 \leq s < \infty \} > -\infty \)? No. It is conceivable that between integers, the reserve process takes much smaller values. But we will show that \( \inf \{ R_s : 0 \leq s < \infty \} > -\infty \).
There are other random walks that are embedded in the reserve process:

**Example 112** Consider the process at claim times $W_n = R_{T_n}$, $n \geq 0$, where $(T_n)_{n \geq 0}$ are the event times of the Poisson process (and $T_0 = 0$). Now

$$W_{n+1} - W_n = R_{T_{n+1}} - R_{T_n} = c(T_{n+1} - T_n) - A_{n+1}, \quad n \geq 0,$$

are also independent identically distributed increments with $E(W_{n+1} - W_n) = c/\lambda - \mu > 0$, and the Strong Law of Large Numbers yields

$$\frac{W_n}{n} = \frac{u}{n} + \frac{1}{n} \sum_{j=1}^{n} (W_{n+1} - W_n) \to c\lambda - \mu \quad \text{a.s. as } n \to \infty.$$

Again, we conclude $W_n \to \infty$, but note that $W_n$ are the local minima of $R$, so

$$I_\infty := \inf \{ R_t : 0 \leq t < \infty \} = \inf \{ W_n, n \geq 0 \} > -\infty.$$

As a consequence, if we denote $R^0_t = ct - C_t$ with associated $I^0_\infty$, then

$$\psi(u) = P_u(R_t < 0 \text{ for some } t \geq 0) = P(I^0_\infty \leq -u) \to P(I^0_\infty = -\infty) = 0$$

as $u \to \infty$, but this is then $\psi(\infty) = 0$.

### 15.3 Ruin probabilities

We now turn to studying the ruin probabilities $\psi(u), u \geq 0$.

**Proposition 113** The ruin probabilities $\psi(u)$ satisfy the renewal equation

$$\psi(x) = g(x) + \int_0^x \psi(x - y) f(y) dy, \quad x \geq 0,$$

where

$$f(y) = \frac{\lambda}{c} \tilde{K}(y) = \frac{\lambda}{c} \int_y^\infty k(x) dx \quad \text{and} \quad g(x) = \frac{\lambda \mu}{c} \tilde{K}_0(x) = \frac{\lambda}{c} \int_x^\infty \tilde{K}(y) dy.$$

**Proof:** Condition on $T_1 \sim \text{Exp}(\lambda)$ and $A_1 \sim k(a)$ to obtain

$$\psi(x) = \int_0^\infty \int_0^x \psi(x + ct - a) k(a) da e^{-\lambda t} dt$$

$$= \int_x^\infty \frac{\lambda}{c} e^{-(s-x)\lambda/c} \int_0^x \psi(s - a) k(a) da ds$$

where we use the convention that $\psi(x) = 1$ for $x < 0$. 
Differentiation w.r.t. $x$ yields

$$
\psi'(x) = \frac{\lambda}{c} \psi(x) - \frac{\lambda}{c} \int_0^\infty \psi(x - a) k(a) da
= \frac{\lambda}{c} \psi(x) - \frac{\lambda}{c} \int_0^x \psi(x - a) k(a) da - \frac{\lambda}{c} K(x).
$$

Note that we also have a terminal condition $\psi(\infty) = 0$. With this terminal condition, this integro-differential equation has a unique solution. It therefore suffices to check that any solution of the renewal equation also solves the integro-differential equation.

For the renewal equation, we only sketch the argument since the technical details would distract from the main steps: note that differentiation (we skip the details for differentiation under the integral sign!) yields (setting $s = x - y$ in the convolution integral)

$$
\psi'(x) = g'(x) + \psi(x) f(0) + \int_0^x \psi(s) f'(x - s) ds
= -\frac{\lambda}{c} K(x) + \frac{\lambda}{c} \psi(x) - \frac{\lambda}{c} \int_0^x \psi(x - a) k(a) da,
$$

and note also that, with the convention $\psi(x) = 1$ for $x < 0$, we can write the renewal equation as

$$
\psi(x) = \int_0^\infty \psi(x - y) f(y) dy,
$$

where $f$ is a nonnegative function with $\int_0^\infty f(y) dy = \lambda/c < 1$, so for any nonnegative solution $\psi \geq 0$, $\psi(x)$ is less than an average of $\psi$ on $(-\infty, x]$, and hence $\psi$ is decreasing (this requires a bit more care), so $\psi(\infty)$ exists with (by monotone convergence using $\psi(x - y) \downarrow \psi(\infty)$ as $x \to \infty$)

$$
\psi(\infty) = \lim_{x \to \infty} \int_0^\infty \psi(x - y) f(y) dy = \int_0^\infty \psi(\infty) f(y) dy = \frac{\lambda}{c} \psi(\infty) \quad \Rightarrow \quad \psi(\infty) = 0.
$$

Example 114 We can calculate $\psi(0) = g(0) = \lambda \mu/c$. In particular, zero initial reserve does not entail ruin with probability 1. In other words, $\psi$ jumps at $u = 0$ from $\psi(0^-) = 1$ to $\psi(0) = \psi(0^+) = \lambda \mu/c < 1$.

Corollary 115 If $\lambda \mu \leq c$, then $\psi$ is given by

$$
\psi(x) = g(x) + \int_0^x g(x - y) u(y) dy \quad \text{where} \quad u(y) = \sum_{n \geq 1} f^{*n}(y).
$$

Proof: This is an application of Exercise A.6.2(c), the general solution of the renewal equation. Note that $f$ is not a probability density for $\lambda \mu < c$, but the results (and arguments) are valid for nonnegative $f$ with $\int_0^\infty f(y) dy \leq 1$.\qed
Where is the renewal process? For \( \lambda \mu < c \), there is no renewal process with interarrival density \( f \) in the strict sense, since \( f \) is not a probability density function. One can associate a defective renewal process that only counts a geometric number of points, and the best way to motivate this is by looking at \( \lambda \mu = c \), where the situation is nicer. It can be shown that the renewal process is counting new minima of \( R \) or \( R^0 \), not in the time parameterisation of \( R^0 \), but in the height variable, i.e.

\[
Y_h = \# \{ n \geq 1 : W_n \in [-h, 0] \text{ and } W_n = \min\{W_0, \ldots, W_n\} \}, \quad h \geq 0,
\]
is a renewal process. Note that \( f \) is the distribution of \( LU \) where \( L \) is a size-biased claim and \( U \sim Unif(0, 1) \). Intuitively, this is, because big claims are more likely to exceed the previous minimal reserve level, hence size-biased \( L \), but the previous level will only be exceeded by a fraction \( LU \), since \( R \) will not be at its minimum when the claim arrives.

So what happens if \( \lambda \mu < c \)? There will only be a finite number of claims that exceed the previous minimal reserve level since now \( R_t \to \infty \), and \( Y \) remains constant for any lower levels of \( h \).

This is not very explicit. To conclude, let us derive more explicit estimates of \( \psi \).

**Proposition 116** Assume that there is \( \alpha > 0 \) such that

\[
1 = \int_0^\infty e^{\alpha y} f(y) dy = \frac{\lambda}{c} \int_0^\infty e^{\alpha y} \bar{K}(y) dy.
\]

Then there is a constant \( C > 0 \) such that

\[
\psi(x) \sim C e^{-\alpha x} \quad \text{as} \quad x \to \infty.
\]

**Proof:** Define a probability density function \( \hat{f}(y) = e^{\alpha y} f(y) \), and \( \hat{g}(y) = e^{\alpha y} g(y) \) and \( \hat{\psi}(x) = e^{\alpha y} \psi(x) \). Then \( \hat{\psi}(x) \) satisfies

\[
\hat{\psi}(x) = \hat{g}(x) + \int_0^x \hat{\psi}(x-y) \hat{f}(y) dy.
\]

The solution (obtained as in Corollary 115) converges by the key renewal theorem:

\[
\hat{\psi}(x) = \hat{g}(x) + \int_0^x \hat{g}(x-y) \hat{u}(y) dy \to \frac{1}{\mu} \int_0^\infty \hat{g}(y) dy =: C \quad \text{as} \quad x \to \infty,
\]

where

\[
\hat{u}(x) = \sum_{n \geq 1} \hat{f}^{*n}(x).
\]

Note that \( \hat{g} \) is not necessarily non-increasing, but it can be checked that it is integrable, and a version of the key renewal theorem still applies.
Example 117 If $A_n \sim \text{Exp}(1/\mu)$, then in the notation of Proposition 113

$$g(x) = \frac{\lambda \mu}{c} e^{-x/\mu} \quad \text{and} \quad f(y) = \frac{\lambda}{c} e^{-y/\mu}$$

so that the renewal equation becomes

$$e^{x/\mu} \psi(x) = \frac{\lambda \mu}{c} + \frac{\lambda}{c} \int_0^x \psi(y) e^{y/\mu} dy.$$

In particular $\psi(0) = \lambda \mu / c$. After differentiation and cancellation

$$\psi'(x) = \left( \frac{\lambda}{c} - \frac{1}{\mu} \right) \psi(x) \quad \Rightarrow \quad \psi(x) = \frac{\lambda \mu}{c} \exp \left\{ - \frac{c - \lambda \mu}{c \mu} x \right\}.$$

15.4 Some simple finite-state-space models

Example 118 (Sickness-death) In health insurance, the following model arises. Let $S = \{H, S, \Delta\}$ consist of the states healthy, sick and dead. Clearly, $\Delta$ is absorbing. All other transitions are possible, at different rates. Under the assumption of full recovery after sickness, the state of health of the insured can be modelled by a continuous-time Markov chain.

Example 119 (Multiple decrement model) A life assurance often pays benefits not only upon death but also when a critical illness or certain losses of limbs, sensory losses or other disability are suffered. The assurance is not usually terminated upon such an event.

Example 120 (Marital status) Marital status has a non-negligible effect for various insurance types. The state space is $S = \{B, M, D, W, \Delta\}$ to model bachelor, married, divorced, widowed, dead. Not all direct transitions are possible.

Example 121 (No claims discount) In automobile and some other general insurances, you get a discount on your premium depending on the number of years without (or at most one) claim. This gives rise to a whole range of models, e.g. $S = \{0\% , 20\% , 40\% , 50\% , 60\\% \}$.

In all these examples, the exponential holding times are not particularly realistic. There are usually costs associated either with the transitions or with the states. Also, estimation of transition rates is of importance. A lot of data are available and sophisticated methods have been developed.