A.6 Renewal theory II

1. (a) Let $p(n) = (n!)^{-1}\lambda^n e^{-\lambda}$ be the probability mass function of the $Poi(\lambda)$ distribution. Calculate the associated size-biased distribution. For a random variable X^{sb} with the size-biased distribution, show that $X^{sb} - 1 \sim Poi(\lambda)$.

   (b) Let $f(x) = (\Gamma(a))^{-1}\lambda^a x^{a-1} e^{-\lambda x}$ be the density of the $Gamma(a, \lambda)$ distribution

   (i) Calculate and identify the associated size-biased distribution.

   (ii) Suppose that the counting process of buses at a particular bus stop can be modelled by a renewal process X with stationary increments and $Gamma(a, \lambda)$ interarrival times. Calculate the average waiting time m_{stat} of a customer arriving at time t.

   (iii) Also calculate the average waiting time m_{ren} of a customer arriving just after a bus has passed. Deduce that

$$m_{stat} > m_{ren} \iff a < 1$$

   This is a version of the waiting time paradox. What is paradoxical here?

2. Let X be a renewal process with continuous interarrival times of finite mean μ and density f.

   (a) Show that $m(t) = \mathbb{E}(X_t)$ can be expressed as

$$m(t) = \sum_{k=1}^{\infty} F_k(t)$$

   where F_k is the distribution function of the kth arrival time T_k and deduce that m is differentiable with

$$m'(t) = \sum_{k=1}^{\infty} f^{*(k)}(t)$$

   where $f^{*(k)}$ is the density of T_k, the kth convolution power of f.

   (b) Show that $m(t)$ satisfies

$$m(t) = F(t) + \int_0^t m(t-x)f(x)dx, \text{ in short: } m = F + m * f.$$

   (c) Show that, for any locally bounded (measurable) function $H : [0, \infty) \to \mathbb{R}$, the function r given by $r = H + H * m'$ satisfies the so-called *renewal-type equation* $r = H + r * f$. MSc students and brave undergraduates should prove uniqueness.

   (d) Show that the excess lifetime $E_t = T_{X_t+1} - t$ has survival function

$$\mathbb{P}(E_t > y) = \bar{F}(t+y) + \int_0^t \bar{F}(t+y-x)m'(x)dx$$

Hint: The convolution operation is a multiplication in the axiomatic sense that $f * g = g * f$, $(f * g) * h = f * (g * h)$, $(f + g) * h = f * h + g * h$ etc.

3. (a) Let X be a renewal process with continuous interarrival times of finite mean μ. Deduce from Exercise 2 and the Key Renewal Theorem that the limiting survival function of the excess life E_t at time t, as $t \to \infty$, is

$$\bar{F}_0(y) = \frac{1}{\mu} \int_y^\infty \bar{F}(z) dz.$$

   (b) Let X be a renewal process with 1-arithmetic (in particular integer-valued) interarrival times Z_j of finite mean μ.

   (i) Show that $E_n = T_{X_n+1} - n$ is a discrete-time Markov chain.

   (ii) Calculate its stationary distribution and deduce that

$$\mathbb{P}(E_n = k) \to \mu^{-1}\mathbb{P}(Z_1 \geq k) \qquad \text{as } n \to \infty.$$

   (iii) Show that $\mu^{-1}\mathbb{P}(Z_1 \geq k)$ is the probability mass function of a random variable U picked uniformly from $\{1, \ldots, S\}$ conditionally given S, where S has the size-biased distribution associated with the distribution of Z_1.

MSc students and keen undergraduates should also try to solve the following exercises.

4. Find the distribution of the excess lifetime for a renewal process each of whose interarrival times is the sum of two independent exponentially distributed random variables having respective parameters α and β. Show that the excess lifetime has mean

$$\frac{1}{\beta} + \frac{\alpha e^{-(\alpha+\beta)t} + \beta}{\alpha(\alpha + \beta)}.$$

Hint: Reformulate in terms of the two-state Markov chain of Exercise A.3.4.

5. Let $\tilde{X}$ be a delayed renewal process whose first arrival time has density g, the subsequent interarrival times density f.

   (a) Show that $\tilde{m}(t) = \mathbb{E}(\tilde{X}_t)$ (as opposed to the undelayed $m(t) = \mathbb{E}(X_t)$) satisfies both

$$\tilde{m} = G + m * g \quad \text{and} \quad \tilde{m} = G + \tilde{m} * f.$$

   (b) Show, now by conditioning on the last arrival before time t that

$$\mathbb{P}(\tilde{E}_t > y) = \bar{G}(t + y) + \int_0^t \bar{F}(t + y - x)\tilde{m}'(x)dx.$$

   (c) If more specifically $g(y) = f_0(y) = \bar{F}(y)/\mu$, show that $\tilde{m}(t) = t/\mu$ and that $\tilde{E}_t$ also has survival function $\bar{F}_0$, for all $t \geq 0$.