A.5 Renewal theory I

1. Potential customers arrive at a single-server bank according to a Poisson process $(N_t)_{t\geq 0}$ with rate λ. However, potential customers will enter the bank only if the server is free when they arrive, and otherwise will go home. Assume that the service times are independent random variables with probability density function g and mean ν.

   (a) Denote by X_t the number of customers that *have left after completed service* before time t, $t \geq 0$. Show that $(X_t)_{t\geq 0}$ is a renewal process, and describe its interarrival distribution.

   (b) Calculate the asymptotic rate $\lim_{t\to\infty} X_t/t$ at which customers leave the bank (after completed service).

   (c) Consider the proportion $P_t = X_t/N_t$. What long-term proportion of potential customers are actually served?

   (d) Consider the sequence of departure times T_n, $n \geq 1$ (departures after completed service). What long-term proportion of time is the server busy? Hint: Consider this proportion at departure times first and then argue as in the proof of the strong law of renewal theory.

2. (a) Let $X, X_1, X_2, \ldots$ be random variables with values in $\mathbb{N}$. Show that, as $n \to \infty$, $X_n \to X$ in distribution if and only if $\mathbb{P}(X_n = k) \to \mathbb{P}(X = k)$ for all $k \in \mathbb{N}$.

   Determine the limiting behaviour in distribution, in probability, almost surely and as convergence of means, of the following sequences of random variables.

   (b) $X_n = \begin{cases} n^3 & \text{with probability } n^{-2} \\ 0 & \text{with probability } 1 - n^{-2} \end{cases}$

   (c) $Y_{2^n+k} = \begin{cases} 1 & \text{if } (k-1)2^{-n} \leq U \leq k2^{-n} \\ 0 & \text{otherwise} \end{cases}$ for $n \geq 0$ and $1 \leq k \leq 2^n$, for a uniform random variable $U \sim Unif(0,1)$. Hint: For almost sure convergence, argue for each possible value $u \in [0,1]$ for U and conclude using $\mathbb{P}(U \in [0,1]) = 1$.

   What is the relevance of these two examples for the implications between the different modes of convergence.

3. Let X be a renewal process whose interarrival times $(Z_n)_{n\geq 0}$ satisfy $0 < \sigma^2 = Var(Z_1) < \infty$ and $\mu = \mathbb{E}(Z_1)$. Deduce from the Central Limit Theorem for $(Z_n)_{n\geq 0}$ that

$$\frac{X_t - t/\mu}{\sqrt{t\sigma^2/\mu^3}} \to Z \sim \mathcal{N}(0,1) \qquad \text{in distribution, as } t \to \infty.$$

   Hint: Express probabilities involving X_t in terms of T_n.

4. Let X be an irreducible positive recurrent continuous-time Markov chain with holding time parameters λ_i and mean passage times m_i. Denote by $H_i^{(m)}$, $m \geq 1$, the successive passage times of X in i.

   (a) Fix $i \in \mathbb{S}$. Show that the increments $Z_m = S_{m+1} - S_m$ of $S_m = H_i^{(m)}$, $m \geq 0$, form a sequence of independent and identically distributed random variables given $X_0 = i$. Hint: Use the strong Markov property at S_m, $m \geq 1$.

   (b) For $i \in \mathbb{S}$ fixed as in (a) and any $j \in \mathbb{S}$, show that given $X_0 = i$,

   $$\frac{S_m}{m} \to m_i = \mathbb{E}(Z_1) \qquad \text{almost surely, as } m \to \infty.$$

   What if $X_0 = j$? Hint: Only the distribution of Z_0 is different now. Consider Z_0 separately.

   (c) Prove the following form of the ergodic theorem.

   $$\frac{1}{t}\int_0^t 1_{\{X_s=i\}} ds \to \frac{1}{m_i\lambda_i} \qquad \text{almost surely, as } t \to \infty.$$

   Hints: Use (b) and also apply the strong law of large numbers to the holding times at i. Consider $t = H_i^{(m)}$, $m \to \infty$, first and deduce the general statement.

   (d) Assume that (c) also holds in L^1 (requires dominated convergence), deduce $1/(\lambda_i m_i) = \xi_i$. Hints: compare (c) with the convergence theorem for continuous-time Markov chains and deduce that the limits must be the same. You may prove and use the fact that $x_t \to x$ implies $\frac{1}{t}\int_0^t x_s ds \to x$.

M.Sc. students and keen undergraduates should also try to solve the following exercise.

5. Let $\mathbb{S} = \mathbb{N} \cup \{\emptyset\}$, and X_t be the number of customers *in the queue* of a single-server queueing system with Poisson arrivals at rate λ and exponential service times at rate μ. The system has the following special feature: the server can serve two customers at the same time. He can also serve a single customer in the system but then a second customer cannot be jointly served before the single customer leaves. Distinguish states 0 (server busy, no queue) and $\emptyset$ (server idle).

   (a) Determine the Q-matrix and the invariant distribution. Hint: Try $\xi_n = \alpha^n \xi_0$ for $n \in \mathbb{N}$.

   (b) Determine the long-term proportion of customers that are served alone. Hint: Which transitions correspond to a beginning single service? Consider the counting processes counting these transitions separately.