A.4 Continuous-time Markov chains II

1. (a) Let $(X_t)_{t\geq 0}$ be a continuous-time Markov chain with finite state space $\mathbb{S}$ and Q-matrix $Q = (q_{i,j})_{i,j\in\mathbb{S}}$. Show that if a distribution $\xi = (\xi_i)_{i\in\mathbb{S}}$ satisfies the *detailed balance equations*

$$\xi_i q_{ij} = \xi_j q_{ji}, \qquad \text{for all } i, j \in \mathbb{S},$$

then ξ is a stationary distribution.

(b) Consider the Q-matrix $Q = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$. What does this Q-matrix tell you about the converse of (a)?

(c) Consider a fleet of N buses. Each bus breaks down independently at rate μ, when it is sent to the depot for repair. The repair shop can only repair one bus at a time and each bus takes an exponential time of parameter λ to repair. Find the equilibrium distribution of the number of buses in service.

2. Consider Markov chains X and Y with Q-matrices on $\{1,2,3,4\}$ and $\{1,2,3,4,5\}$

$$Q_X = \begin{pmatrix} -1 & 1/2 & 1/2 & 0 \\ 1/4 & -1/2 & 0 & 1/4 \\ 1/6 & 0 & -1/3 & 1/6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Q_Y = \begin{pmatrix} -3 & 2 & 0 & 0 & 1 \\ 0 & -3 & 3 & 0 & 0 \\ 0 & 5 & -5 & 0 & 0 \\ 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

(a) For X, calculate the expected time to hit 4 starting from 1

(b) For X, calculate the probability of hitting 3 starting from 1.

(c) For Y, determine all stationary distributions.

(d) For Y, determine the limit distribution when starting from 1.

Hint: For (a) and (b) you may wish to consider the quantities for arbitrary starting points and derive linear equations by conditioning on the first transition (time and state).

3. Consider a single-server queue in which customers arrive in a Poisson process of rate λ and service times are independent identically exponentially distributed with parameter μ. Let $X(t)$ denote the length of the queue at time t including any customer being served, where $X(0) = 0$.

(a) If the queue length is 1, what is the probability that the next customer arrives before the current customer's service time ends?

(b) Set up the detailed balance equations and deduce the invariant distribution of X, stating the condition under which it holds. What can you deduce from the detailed balance equations when the condition fails?

(c) Briefly describe the behaviour of the 'jump chain' M formed by considering the sequence of values of X. Find the invariant distribution of the jump chain.

(d) Formulate the ergodic theorems for X and M. Use this to explain why the invariant distributions in (b) and (c) are different.

(e) What is the distribution of the number N of customers arriving during a given service period? Hint: Set $p = \mathbb{P}(N = 0)$ and first calculate $\mathbb{P}(N = n|N \geq n)$.

M.Sc. students and keen undergraduates should also try to solve the following exercise.

4. A *non-minimal* continuous-time Markov chain. Consider the explosive birth process with rates $\lambda_n = 2^n$, $n \geq 0$. Denote its Q-matrix by Q. Let $Y_n^{(m)} \sim Exp(\lambda_n)$, $n \geq 0$, $m \geq 1$, be independent. Define a process X taking values in $\mathbb{N}$ by

$$X_t = n \qquad \text{if } T_n^{(m)} \leq t < T_{n+1}^{(m)},$$

where $T_\infty^{(0)} = 0$ and $T_n^{(m)} = T_\infty^{(m-1)} + \sum_{k=0}^{n-1} Y_k^{(m)}$, $n \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, $m \geq 1$. Note that this just means that the birth process starts afresh from 0 after explosion. You may assume that this defines X_t for all $t \in [0, \infty)$, with probability 1. (This can be proved using the Strong Law of Large Numbers, cf. later in the course).

(a) Calculate all non-negative solutions ξ of $\xi Q = 0$.

(b) For $i \in \mathbb{N}$, let $H_i = \inf\{t > T_1 : X_t = i\}$ denote the first hitting time of i. Calculate $m_i = \mathbb{E}_i(H_i)$ and define $\xi_i = 1/(m_i\lambda_i)$. Note that ξ is a probability distribution. Comments! Hint: what is the relationship between $T_\infty^{(1)}$ and H_0?

Please read and comment on the following first, then prove as far as possible that

(c) given $X_0 = i$, the hitting time $H_j^{(1)} = H_j$ of $j > i$, has density function

$$f_{ij}^{(1)} = f_{ij} = \exp_{\lambda_i} * \ldots * \exp_{\lambda_{j-1}} \qquad \text{and vice versa } f_{ji} \text{ such that } f_{ij} * f_{ji} = f,$$

where $\exp_\lambda$ denotes the density of $Exp(\lambda)$, $*$ denotes convolution $g * h(x) = \int_0^x g(y)h(x-y)dy$, and $f = f_{ii} = f_{00}$ is the density of T_∞ (which you may assume to exist, as well as f_{ji}). The densities of successive hitting times $H_j^{(m+1)}$, $m \geq 1$, are then

$$f_{ij}^{(m+1)} = f_{ij} * f^{*(m)}$$

where $\cdot^{*(m)}$ denotes mth convolution power $f^{*(1)} = f$ and $f^{*(m+1)} = f * f^{*(m)}$. Hint: remember that the density of the sum of continuous random variables is the convolution product of their densities.

(d) Show that for all $t \geq 0$ and $i, j \in \mathbb{N}$

$$p_{ij}(t) := \mathbb{P}_i(X_t = j) = \frac{1}{\lambda_j} \sum_{m \geq 1} f_{i,j+1}^{(m)}(t).$$

(e) Show that $\xi P(t) = \xi$ for all $t \geq 0$. Comments!

And for the tireless:

(f) Show that $QP(t) = P'(t)$. Comments! What about $P(t)Q$?
Hint: $(g * h)'(x) = g * (h')(x) + g(x)h(0) = (g') * h(x) + g(0)h(x)$.