A.3 Continuous-time Markov chains

1. Explain the evolution of a continuous-time Markov chain with Q matrix
$$Q = \begin{pmatrix} -4 & 2 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 3 & 0 & -5 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
In particular, write down holding rates and transition probabilities of the jump chain, and draw a typical path.

2. Customers arrive at a store at the instants of a Poisson process of rate 2. At the door, two representatives separately demonstrate the same product to anybody entering the store. Each demonstration takes a time which is exponentially distributed with parameter 1, and is independent of other demonstrations. After the demonstration the customer enters the store. If both representatives are busy the customer goes directly into the store.

   If both representatives are free at $t = 0$, show that the probability that both are busy at $t > 0$ is
$$\frac{2}{5} - \frac{2}{3}e^{-2t} + \frac{4}{15}e^{-5t}.$$
   Hint: You don't want to count customers *in* the shop. What is your Markov chain? Show that the process described is indeed a continuous-time Markov chain.

3. Let X be a simple birth-death process where individuals have independent $Exp(\mu)$ lifetimes and, during their lifetime give birth at rate λ independently of other individuals. Suppose that $X_0 = 1$. Let
$$G(s,t) = \mathbb{E}(s^{X_t}) \qquad \text{and} \qquad m(t) = \mathbb{E}(X_t)$$
   (a) Show that the simple birth-death process is a continuous-time Markov chain. Construct the Q matrix and write down the forward equations.

   (b) Show that G satisfies
$$\frac{\partial}{\partial t}G(s,t) = (\lambda s - \mu)(s-1)\frac{\partial}{\partial s}G(s,t).$$
   (c) Deduce that $m'(t) = (\lambda - \mu)m(t)$.

   (d) For $\lambda \neq \mu$ it can be shown that
$$\mathbb{E}\left(s^{X_t}\right) = \frac{\mu(s-1) - (\lambda s - \mu)e^{-(\lambda-\mu)t}}{\lambda(s-1) - (\lambda s - \mu)e^{-(\lambda-\mu)t}}, \tag{1}$$
   for $\lambda = \mu$
$$\mathbb{E}\left(s^{X_t}\right) = \frac{\lambda t(s-1) - s}{\lambda t(s-1) - 1}. \tag{2}$$
   Let T be the first time at which X_t takes the value zero. Find the distribution function, density and expectation of T when $\lambda \leq \mu$. Hint: $G(0,t) = \mathbb{P}(X_t = 0)$.

   (e) Using (1), (2) and the continuity theorem for probability generating functions, identify the limiting distribution of X_t as $t \to \infty$. Hint: $\mathbb{P}(X_\infty \in \{0, \infty\}) = 1$.

4. Consider the continuous-time Markov chain with Q-matrix

$$Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}.$$

   (a) Write down the backward and forward equations. Solve either the backward or the forward equations for the transition probabilities $p_{ij}(t)$, $i, j = 1, 2$. Check that your solution also satisfies the other equations. Hint: one set is easier to solve than the other: the solutions of $y' = a + by$ are $y(x) = ce^{bx} - a/b$, $c \in \mathbb{R}$.

   (b) Calculate Q^n by setting up recurrence relations for its entries (or otherwise) and hence find

$$e^{tQ} := \sum_{n=0}^{\infty} \frac{t^n Q^n}{n!}.$$

   Compare your answer with that to part (a).

   (c) Solve the equation $\xi Q = 0$ for ξ and verify that $p_{ij}(t) \to \xi_j$ as $t \to \infty$. What is the interpretation of this?

M.Sc. students and keen undergraduates should also try and solve the following exercise.

5. Let X be a continuous-time Markov chain in a finite state space $\mathbb{S}$, with Q-matrix Q and transition matrices $P(t)$, $t \geq 0$.

   (a) Show that for all $i, j \in \mathbb{S}$, as $h \downarrow 0$

$$p_{ij}(h) = \delta_{ij} + q_{ij} h + o(h) \tag{3}$$

   where δ_{ij} is defined as 1 if $i = j$ and 0 otherwise. *Hint:* Derive lower bounds by restricting to the event that only one jump occurs before time h. Deduce upper bounds by summing over $j \in \mathbb{S}$.

   Remark: It can be shown that a right-continuous process $(X_t)_{t \geq 0}$ on $\mathbb{S}$ is a continuous-time Markov chain if and only if X_{t+h} is conditionally independent of $(X_r)_{r \leq t}$ given $X_t = i$ (Markov property) and $P(X_{t+h} = j | X_t = i) = \delta_{ij} + q_{ij} h + o(h)$ uniformly in t.

   (b) Show that, as $h \downarrow 0$,

$$\frac{p_{ik}(t+h) - p_{ik}(t)}{h} = \sum_{j \in \mathbb{S}} p_{ij}(t) q_{jk} + o(1)$$

   and deduce that $P(t)$, $t \geq 0$, satisfies the forward equation $P'(t) = P(t)Q$, $P(0) = I$.

   (c) Generalise the setting in (a) and (b) to countably infinite $\mathbb{S}$ and minimal continuous-time Markov chains. Suggest conditions under which the arguments still work. There are in fact other arguments, that allow to establish the result without extra conditions.