A.1 Poisson processes and conditioning

1. (a) Let Y be an exponential random variable with parameter $\lambda \in (0, \infty)$ modelling a lifetime. Show the *simple lack of memory property* that at all times $t \in [0, \infty)$, the residual lifetime $Y - t$ given survival $\{Y > t\}$ is again exponential with parameter λ, i.e. that $\mathbb{P}(Y - t > s | Y > t) = e^{-\lambda s}$ for all $s \in [0, \infty)$.

   (b) Converse: let Z be any random variable taking values in $[0, \infty)$ whose distribution has the lack of memory property (i.e. the conditional distribution of the residual lifetime given survival to time $t \in [0, \infty)$ does not depend on t). By studying survival probabilities $\mathbb{P}(Z > s)$ (or cumulative distribution functions), show that Z is exponential with some parameter $\mu \in (0, \infty)$.

   (c) Let Y be exponential with parameter λ. From the definition of conditional probabilities $\mathbb{P}(A|B)$, find the conditional distribution functions (choose $A = \{Y \leq s\}$, or survival functions, $A = \{Y > s\}$), densities and expectations of

      i. Y given $B_1 = \{Y \leq t\}$,

      ii. Y given $B_2 = \{Y > t\}$.

   (d) Let $(X_t)_{t \geq 0}$ be a Poisson process with rate λ. Show that conditionally on $(X_t)_{t \geq 0}$ having exactly one jump in the interval $[0, t]$, the jump time is uniformly distributed on $[0, t]$. Hint: Calculate $\mathbb{P}(Z_1 \leq r | X_t = 1)$ where Z_1 is the first jump time, using the Properties of the Poisson process.

2. Let X and Y be independent exponential random variables (competing exponential alarm clocks) with respective parameters λ and μ. Let

$$W = \min\{X, Y\}, \ Z = \max\{X, Y\}, \ O = Z - W, \ M = 1_{\{X \leq Y\}} = \begin{cases} 1 & \text{if } X \leq Y, \\ 0 & \text{if } X > Y. \end{cases}$$

   (a) Calculate $\mathbb{P}(W > s)$ and $\mathbb{P}(M = 1)$. Identify the distributions of W and M.

   (b) Express then event $\{W \leq w, M = 1, O \leq t\}$ in terms of X and Y and calculate its probability. What is $\mathbb{P}(W \leq w, M = 0, O \leq t)$?

   (c) Are W and (M, O) independent? You may assume without proof that it is enough to check that $\mathbb{P}(W \in A, (M, O) \in B) = \mathbb{P}(W \in A)\mathbb{P}((M, O) \in B)$ for $A = [0, w]$ and $B = \{m\} \times [0, t]$ for all $w \geq 0$, $m \in \{0, 1\}$ and $t \geq 0$.

   (d) Are M and O independent? Find the conditional distributions of O given $M = 0$, and of O given $M = 1$. Is this related to the *lack of memory property*?

3. Queueing models consist of arrival and service of customers. At this point, we leave the two parts separate, so (a) and (b) are unrelated.

   (a) A bank has two clerks. Service times at this bank are independent and exponentially distributed with parameter μ. When the bank opens at 9am, you enter the bank together with two other customers. You are generous and let the other two customers proceed to the two clerks. You will then be the next to be served by the next available clerk; what is the probability that, of the three customers, you will be the last to leave? Hint: Express the event in question in terms of three independent random variables $X, Y, Z \sim Exp(\mu)$.

(b) Passengers arrive at a bus stop in a Poisson process of rate one per minute. A bus arrives and clears the queue. If the time T until the next bus arrives is uniformly distributed on the interval $(10, 15)$ minutes and N is the number of people in the queue when it does arrive, find $\mathbb{E}(N)$ and $Var(N)$. (You may assume that nobody leaves the queue once they have joined it.) What is the probability that there are no customers in the queue when the bus arrives?

M.Sc. students and keen undergraduates should also try and solve the following exercises. There will be such exercises on all assignment sheets. They will enhance your understanding, give some more practice and clarify some technical points in the lectures.

4. (a) Generalise Exercise 2 to n competing exponential clocks, study their minimum and the resulting overshoots (residual times until the remaining clocks ring).

   (b) Let X be an exponential random variable and R an independent nonnegative random variable. Show that X has the *lack of memory property* also at the random time R, i.e.

$$\mathbb{P}(X > R + u | X > R) = \mathbb{P}(X > u).$$

   Hint: Calculate $\mathbb{P}(X > R + u)$ by conditioning on X.

5. Let N be a random variable taking values in a countable set $\mathbb{S}$. Let Z be a random variable whose conditional distribution given $\{N = s\}$ does not depend on $s \in \mathbb{S}$.

   (a) Show that $\mathbb{P}(Z \in A, N = s) = \mathbb{P}(Z \in A)\mathbb{P}(N = s)$ for all (measurable) sets A and $s \in \mathbb{S}$, i.e. that N is independent of Z.

   (b) Let M be a random variable that is conditionally independent of Z given N. Show that (M, N) is independent of Z.

6. Let T_n be the time of the nth arrival in a Poisson process X with rate λ, and define the excess lifetime process $E = (E_t)_{t \geq 0}$ by

$$E_t = T_{X_t + 1} - t.$$

   This is the time one must wait after time t before the next arrival.

   (a) Sketch a realisation of X illustrating E_t for a fixed $t > 0$. Show E in a second sketch underneath.

   (b) Show by conditioning on T_1 that

$$\mathbb{P}(E_t > x) = e^{-\lambda(t+x)} + \int_0^t \mathbb{P}(E_{t-u} > x)\lambda e^{-\lambda u} du.$$

   Hints: $\mathbb{P}(E_s > x) = \mathbb{E}(1_{\{E_t > x\}})$ where $1_{\{E_t > x\}}$ is the random variable that is 1 if $E_t > x$ and 0 otherwise. You may wish to rewrite $\{T_1 = u\}$ so that you can apply the Markov property of X.

   (c) Solve this integral equation to find the survival function of E_t.

   (d) How does this relate to the *lack of memory property*?