4 Reserving and further life insurance products (incomplete!)

A.1. Let T_x and T_y be two independent residual lifetimes and i a constant interest rate. Describe the benefit which has the present value random variable given by Z below:

$$Z = \begin{cases} 0 & T_x < T_y \\ \overline{a}_x - \overline{a}_y & T_x \geq T_y. \end{cases}$$

A.2. Consider a constant interest rate i and a multiple decrement model. Express in terms of the hazard rates $\mu_x^{(j)}$, $x \geq 0$, of causes $1 \leq j \leq m$ the expected present value of a life assurance that pays 1 only on death due to cause 1.

A.3. Consider the interest model of an effective annual rate of $i > 0$.

(i) Consider a lifetime random variable K with geometric distribution

$$\mathbb{P}(K = k) = (1-q)^{k-1}q, \qquad k \geq 1.$$

Show that the price of a whole-life annuity is $(1-q)/(i+q)$.

(ii) Consider a discounted dividend model, in which dividend payments D_k, $k \geq 1$, are such that $\mathbb{E}(D_k) = (1+g)^k$, $k \geq 1$. Show that the total expected present value of all dividend payments is $(1+g)/(i-g)$.

(iii) Consider a discounted dividend model for company shares, in which the company has a geometrically distributed insolvency time K. Dividends satisfy $\mathbb{E}(D_k|K \geq k) = (1+h)^k$. Show that the total expected present value of all dividend payments is $(1+h)(1-q)/(i+q-h(1-q))$.

B.1. Some time ago, a life office issued an assurance policy to a life now aged exactly 55. Premiums are payable annually in advance, and death benefits are paid at the end of the year of death. The office calculates reserves using gross premium policy values. The following information gives the reserve assumptions for the policy year just completed. Expenses are assumed to be incurred at the start of the policy year.

Reserve brought forward at the start of the policy year: £12,500

Annual premium: £1,150

Annual expenses: £75

Death benefit: £50,000

Mortality: A1967/70

Interest 5.5% per annum

Calculate the reserve at the end of the policy year.

B.2. (i) Show that $\overline{A}_x = 1 - \delta\overline{a}_x$.

(ii) Consider a whole life assurance with sum assured 1 payable at the point of death, with a constant premium paid continuously. Show that the reserve at time t satisfies

$${}_t\overline{V}_x = 1 - \frac{\overline{a}_{x+t}}{\overline{a}_x}$$

B.3. An insurer issues n identical policies. Let Y_j be the claim amount from the jth policy, and suppose that the random variables Y_j, $j = 1, \ldots, n$ are i.i.d. with mean $\mu > 0$ and variance σ^2. The insurer charges a premium of A for each policy.

(a) Show that if $A = \mu + 10\sigma n^{-1/2}$, then the probability that total claims exceed total premiums is no more than 1%, for any value of n.

(b) Use the Central Limit Theorem to show that if instead $A = \mu + 3\sigma n^{-1/2}$, then this probability is still less than 1%, provided n is large enough.

B.4 Consider the constant interest rate model at rate i, and consider m independent lifetime random variables $T^{(j)}$, with respective force of mortality $\mu_t^{(j)}$, $t \geq 0$, $1 \leq j \leq m$. Suppose that the forces of mortality are constant equal to $\mu_{x+1/2}^{(j)}$ for $x \leq t < x+1$. Let $T_{\min} = \min\{T^{(1)}, \ldots, T^{(m)}\}$ be the time of the first death and $J = j$ if $T_{\min} = T^{(j)}$, $1 \leq j \leq m$. Let $K^{(j)} = [T^{(j)}]$, $1 \leq j \leq m$, the associated curtate lifetimes.

(a) Suppose that the $T^{(j)}$ are lives aged x_j, $1 \leq j \leq m$, with common force of mortality μ_t, $t \geq 0$. Express the force of mortality $\mu^{(j)}$ of $T^{(j)}$ in terms of μ.

(b) Consider a life assurance that pays S_j at time T_{rmmin} if $J = j$. Determine the expected discounted value of this assurance in terms of $\mu^{(k)}$, $1 \leq k \leq m$.

(c) Consider a life assurance that pays S_j at $[T_{\min}]+1$ if $J = j$. Determine the expected discounted value of this assurance in terms of $\mu^{(k)}$, $1 \leq j \leq m$, and show that this answer can be written in the form $\sum_{n \geq 0} v^{n+1}{}_npq_n^{(j)}$, where you should explain notation.

(d) Let $m = 2$ in the setting of (a). Consider two life assurances that pay, respectively,

- 1 at the end of the year of the first death,
- 1 at the end of the year of the second death.

Calculate their expected present values A_{xy} and $A_{\overline{xy}}$ in terms of one-year death probabilities q_x, $x \geq 0$. Show that $A_{\overline{xy}} = A_x + A_y - A_{xy}$.

B.5 In a particular accumulation fund income is retained and used to increase the value of the fund unit. The 'middle price' of the unit on 1 April in each of the years 1999 to 2005 is given in the following table:

Year 1 April	1999	2000	2001	2002	2003	2004	2005
Middle price of unit in £	1.86	2.11	2.55	2.49	2.88	3.18	3.52

(a) On the basis of the above prices and ignoring charges:

(i) Show that the yield obtained by an investor who purchased 200 units on 1 April in each year from 1999 to 2004 inclusive, and who sold his holding on 1 April 2005, is approximately 10.60%.

(ii) Show that the yield obtained by a person who invested £500 in the fund on 1 April each year from 1999 to 2004 inclusive, and who sold back his holding to the fund managers on 1 April 2005, is approximately 10.67% (You should assume that investors may purchase fractional parts of units.)

(b) Suppose that, in order to allow for expenses, the fund's managers sell units 2% above the published middle price and buy back units 2% below the middle price. On this basis find revised answers to (i) and (ii) of (a).

to be continued...