S.4 Insurance and Saving

A.1. (a) Denoting the loss incurred by each member of a pool of size 2 and 3 by $\tilde{x}$ and $\tilde{y}$ respectively the pdf of these binomial random variables are:

- $P[\tilde{x} = 0] = (1 - p)^2$
- $P[\tilde{x} = 100/2] = 2p(1 - p)$
- $P[\tilde{x} = 200/2] = p^2$
- $P[\tilde{y} = 0] = (1 - p)^3$
- $P[\tilde{y} = 100/3] = 3p(1 - p)^2$
- $P[\tilde{y} = 200/3] = 3p^2(1 - p)$
- $P[\tilde{y} = 300/3] = p^3$

(b) Let the distribution of $\tilde{\varepsilon}$ conditional on outcomes of $\tilde{y}$ be as follows

- $(\tilde{\varepsilon}|\tilde{y} = 0) = (0, 1)$
- $(\tilde{\varepsilon}|\tilde{y} = 100/3) = (-\frac{100}{3}, \frac{1}{3}; \frac{50}{3}, \frac{2}{3})$
- $(\tilde{\varepsilon}|\tilde{y} = 200/3) = (-\frac{50}{3}, \frac{2}{3}; \frac{100}{3}, \frac{1}{3})$
- $(\tilde{\varepsilon}|\tilde{y} = 300/3) = (0, 1)$

Then $E[\tilde{\varepsilon}|\tilde{y}] = 0$ for any outcome of $\tilde{y}$. Moreover, $\tilde{y} + \tilde{\varepsilon}$ has the same three possible outcomes as $\tilde{x}$ and these have the same probabilities:

- $P[\tilde{y} + \tilde{\varepsilon} = 0] = P[\tilde{y} = 0 \cap \tilde{\varepsilon} = 0] + P[\tilde{y} = 100/3 \cap \tilde{\varepsilon} = -100/3]$
  $= (1 - p)^3 \times 1 + 3p(1 - p)^2 \times \frac{1}{3}$
  $= (1 - p)^2 \times (1 - p + p) = (1 - p)^2 = P[\tilde{x} = 0]$
- $P[\tilde{y} + \tilde{\varepsilon} = 50] = P[\tilde{y} = 100/3 \cap \tilde{\varepsilon} = 50/3] + P[\tilde{y} = 200/3 \cap \tilde{\varepsilon} = -50/3]$
  $= 3p(1 - p)^2 \times \frac{2}{3} + 3p^2(1 - p) \times \frac{2}{3}$
  $= 2p(1 - p)(1 - p + p) = 2p(1 - p) = P[\tilde{x} = 50]$
- $P[\tilde{y} + \tilde{\varepsilon} = 100] = P[\tilde{y} = 100 \cap \tilde{\varepsilon} = 0] + P[\tilde{y} = 200/3 \cap \tilde{\varepsilon} = 100/3]$
  $= p^3 \times 1 + 3p^2(1 - p) \times \frac{1}{3}$
  $= p^2(p + 1 - p) = p^2 = P[\tilde{x} = 100]$

So any risk averse individual with starting wealth $w$, will prefer a pool with $N = 3$ to one with $N = 2$.

B.1. The agent’s expected utility on wagering $w$ is

$$f(w) = pu(X - w + 2w) + (1 - p)u(X - w)$$
$$= -pe^{-r(X+w)} - (1 - p)e^{-r(X-w)}$$
where

\[ f'(w) = rpe^{-r(X+w)} - r(1 - p)e^{-r(X-w)} \]
\[ f''(w) = -r^2pe^{-r(X+w)} - r^2(1 - p)e^{-r(X-w)} < 0 \]

Since \( f \) is twice differentiable with \( f''(w) < 0 \) for all \( w \), the optimal choice of \( w \), \( w^* \), will solve \( f'(w^*) = 0 \).

\[ rpe^{-r(X+w^*)} - r(1 - p)e^{-r(X-w^*)} = 0 \]
\[ \therefore \frac{p}{1-p} = e^{r(X+w^*) - r(X-w^*)} \]
\[ \therefore w^* = \frac{1}{2r} \ln\left(\frac{p}{1-p}\right) \]

B.2. (a) Working in units of £1 million, the premium is \( \frac{1}{4} \times 8 \times (1 + 0.2) = 2.4 \).

(b) Denoting the expected utility on choosing coinsurance level \( \beta \) by \( f(\beta) \), we have

\[ f(\beta) = \frac{3}{4} \ln(12 - 2.4\beta) + \frac{1}{4} \ln(12 - 2.4\beta - 8(1 - \beta)) \]
\[ f'(\beta) = -\frac{3}{4} \frac{2.4}{12 - 2.4\beta} + \frac{1}{4} \frac{5.6}{4 + 5.6\beta} \]
\[ f''(\beta) = -\frac{3}{4} \frac{2.4^2}{(12 - 2.4\beta)^2} - \frac{1}{4} \frac{5.6^2}{(4 + 5.6\beta)^2} < 0 \]

So, \( f \) is twice differentiable and strictly concave, and so the optimal \( \beta \), \( \beta^* \) solves \( f'(\beta^*) = 0 \) (or is at an endpoint \( \beta^* = 0 \) or \( \beta^* = 1 \):

\[ -\frac{3}{4} \frac{2.4}{12 - 2.4\beta^*} + \frac{1}{4} \frac{5.6}{4 + 5.6\beta^*} = 0 \]
\[ \therefore 7.2(4 + 5.6\beta^*) = 5.6(12 - 2.4\beta^*) \]
\[ \therefore \beta^* = \frac{5.6 \times 12 - 7.2 \times 4}{7.2 \times 5.6 + 5.6 \times 2.4} = \frac{38.4}{53.76} = 0.714 \text{ (3 s.f.)} \]

Since this \( \beta^* \in [0, 1] \) and \( f \) is concave, the optimal solution will be \( \beta^* \), not 0 or 1. (We will confirm this in the next part of the question.)

(c)

\[ f(0) = \frac{3}{4} \ln(12) + \frac{1}{4} \ln(4) = 2.210 \text{ (4 s.f.)} \]
\[ f(\beta^*) = \frac{3}{4} \ln \left(12 - 2.4 \frac{38.4}{53.76}\right) + \frac{1}{4} \ln \left(4 + 5.6 \frac{38.4}{53.76}\right) = 2.268 \text{ (4 s.f.)} \]
\[ f(1) = \ln(12 - 2.4) = 2.262 \text{ (4 s.f.)} \]

(d) Full insurance (\( \beta^* = 1 \)) is optimal if loading is zero (Mossin’s Theorem).
B.3. For the standard portfolio problem let the utility function be \( u \), satisfying \( u' > 0 \), \( u'' < 0 \), and initial wealth be \( w_0 \). Since the risk free (risky) investment earns return zero (\( \tilde{x} \)) an investment in the risky asset of \( \alpha \) leads to realised wealth of \( w_0 + \alpha \tilde{x} \) and expected utility, \( f(\alpha) \), of:

\[
f(\alpha) := E[u(w_0 + \alpha \tilde{x})]
\]

\( f \) is strictly concave and so the unique optimal \( \alpha^* \) is the solution to the first order condition

\[
f'(\alpha^*) = E[\tilde{x}u'(w_0 + \alpha^* \tilde{x})] = 0.
\]

(a) Now, let \( \tilde{y} := (\tilde{x}, q; 0, 1 - q) \), and let \( \beta \) denote the optimal amount of (new) asset \( \tilde{y} \) to purchase. The new first order condition is

\[
E_{\tilde{y}}[\tilde{y}u'(w_0 + \beta^* \tilde{y})] = 0
\]

Applying the law of total expectation (\( E_Y(Y) = E_X(E_Y(Y|X)) \)) we find

\[
E_{\tilde{x}}[\tilde{y}u'(w_0 + \beta^* \tilde{y})] = E_{\tilde{x}}[E_{\tilde{x}}[\tilde{y}u'(w_0 + \beta^* \tilde{y})|\tilde{x}]]
\]

\[
= E_{\tilde{x}}[q\tilde{x}u'(w_0 + \beta^* \tilde{x}) + (1 - q) \times 0]
\]

\[
= qE_{\tilde{x}}[\tilde{x}u'(w_0 + \beta^* \tilde{x})]
\]

Now the unique solution to this is \( \beta^* = \alpha^* \), that is you purchase the same amount of the risky asset as you would have in the original problem. Note that this comes from the Independence Axiom: you prefer \( w_0 + \alpha^* \tilde{x} \) to any other \( w_0 + \alpha \tilde{x} \) and therefore prefer any stochastic mix of \( w_0 + \alpha^* \tilde{x} \) and the risk-free asset to any stochastic mix of \( w_0 + \alpha \tilde{x} \) and the risk-free asset, and so \( \alpha^* \) is the optimal amount to purchase in both problems.

(b) Let \( \tilde{z} := q\tilde{x} \), and let \( \gamma \) denote the optimal amount of (new) asset \( \tilde{z} \) to purchase. The new first order condition is

\[
E_{\tilde{z}}[\tilde{z}u'(w_0 + \gamma^* \tilde{z})] = 0
\]

\[
E_{\tilde{z}}[q\tilde{x}u'(w_0 + \gamma^* q\tilde{x})] = 0
\]

From the definition of \( \alpha^* \), we have \( \gamma^* q = \alpha^* \), and therefore \( \gamma^* = \frac{\alpha^*}{q} \).

B.4. The question did not specify whether \( \alpha \) can be negative, or whether the expected excess return from the risky asset was positive. We don’t impose any restrictions in this solution.

(a) Let \( w_0 \) denote initial wealth and \( \alpha \) denote the demand for the risky asset. The optimisation problem is

\[
\max_{\alpha} p \ln(w_0(1 + r) - \alpha(r - a)) + (1 - p) \ln(w_0(1 + r) + \alpha(b - r))
\]
This is strictly concave in $\alpha$ and so the optimal choice of $\alpha$, $\alpha^*$ satisfies first order condition

$$-\frac{p(r-a)}{w_0(1+r) - \alpha^*(r-a)} + \frac{(1-p)(b-r)}{w_0(1+r) + \alpha^*(b-r)} = 0$$

$$.p(r-a)(w_0(1+r) + \alpha^*(b-r)) = (1-p)(b-r)(w_0(1+r) - \alpha^*(r-a))$$

\[\alpha^*[p(r-a)(b-r) + (1-p)(r-a)(b-r)] = w_0(1+r)[(1-p)(b-r) - p(r-a)]\]

$$.\alpha^* = w_0(1+r) \left[ \frac{1-p}{r-a} - \frac{p}{b-r} \right]$$

= $w_0(1+r) \left[ \frac{(1-p)(b-r) + p(a-r)}{(r-a)(b-r)} \right]$

(Note that $\alpha^*$ is the same sign as the excess expected return of the risky asset over the risk-free asset.)

(b) Yes, for $a < r < b$.

(c)  

$$\frac{d\alpha^*}{dw_0} = \frac{\alpha^*}{w_0}$$

So a small increase in initial wealth increases the magnitude of $\alpha^*$; if $\alpha^*$ is positive it becomes more positive, if $\alpha^*$ is negative it becomes more negative.

(d)  

$$\frac{d\alpha^*}{dr} = \frac{\alpha^*}{1+r} - w_0(1+r) \left[ \frac{1-p}{(r-a)^2} + \frac{p}{(b-r)^2} \right]$$

An increase in the risk free rate of return has two effects on the optimal choice of $\alpha$. First there is a ‘wealth’ effect of $\frac{\alpha^*}{1+r}$, whereby by increasing the risk-free rate of return effectively increases the ex-post wealth of the individual, increasing the magnitude of $\alpha$. Second by making the risky asset relatively less attractive it reduces $\alpha$.

(e)  

$$\frac{d\alpha^*}{da} = w_0(1+r) \left[ \frac{1-p}{(r-a)^2} \right] \geq 0$$

$$\frac{d\alpha^*}{db} = w_0(1+r) \left[ \frac{p}{(b-r)^2} \right] \geq 0$$

An increase (decrease) in $a$ or $b$ leads to a first order stochastic dominant improvement (deterioration) in the returns from the risky asset, thereby increasing (decreasing) optimal demand.

(f) Let $c := (1-p)(b-r) + p(a-r)$ and so $r-a = \frac{(1-p)(b-r)-c}{p}$. Then

$$\alpha^* = w_0(1+r) \left[ \frac{p}{b-r - \frac{c}{1-p}} - \frac{p}{b-r} \right]$$
Differentiating $\alpha^*$ with respect to $b$ whilst holding expected excess return $c$ constant gives

$$\frac{\partial \alpha^*}{\partial b} = w_0(1 + r)p \left[ -\frac{1}{(b - r - \frac{c}{1-p})^2} + \frac{1}{(b - r)^2} \right]$$

Since $b - r > 0$ and $r - a > 0$, and therefore $b - r - \frac{c}{1-p} > 0$ we have $\frac{\partial \alpha^*}{\partial b}$ has the opposite sign to $c$, and therefore the opposite sign to $\alpha^*$. So an increase in $b$ whilst holding the expected excess return of the risky asset constant reduces the magnitude of $\alpha^*$.

B.5. (a) The budget constraint is $1 = v c_1 + v^2 c_2$ and so, substituting the budget constraint into the objective function, Sally’s optimisation problem may be written as

$$\max_{c_1} \frac{p_1}{1 + \delta} \ln[c_1] + \frac{p_2}{(1 + \delta)^2} \ln[(1 + i)^2 - (1 + i)c_1]$$

This is strictly concave in $c_1$ and so the optimal $c_1$ satisfies

$$\frac{p_1}{(1 + \delta)c_1^*} - \frac{p_2}{(1 + \delta)^2((1 + i) - c_1^*)} = 0$$

$$\therefore p_1(1 + \delta)((1 + i) - c_1^*) = p_2 c_1^*$$

$$\therefore c_1^* = \frac{p_1(1 + \delta)(1 + i)}{p_2 + p_1(1 + \delta)} = 0.741 \text{ (3 s.f.)}$$

$$c_2^* = (1 + i)^2 - (1 + i)c_1^* = 0.395 \text{ (3 s.f.)}$$

$$\therefore U(c_1^*, c_2^*) = \frac{p_1}{1 + \delta} \ln[c_1^*] + \frac{p_2}{(1 + \delta)^2} \ln[c_2^*] = -0.512 \text{ (3 s.f.)}$$

(b) Now the budget constraint is $1 = p_1 v c_1 + p_2 v^2 c_2$ and the optimisation problem is

$$\max_{c_1} \frac{p_1}{1 + \delta} \ln[c_1] + \frac{p_2}{(1 + \delta)^2} \ln\left[\frac{(1 + i)^2 - p_1(1 + i)c_1}{p_2}\right]$$

which is also strictly concave and has first order condition

$$\frac{p_1}{(1 + \delta)c_1^{**}} - \frac{p_2 p_1}{(1 + \delta)^2((1 + i) - p_1 c_1^{**})} = 0$$

$$\therefore (1 + \delta)((1 + i) - p_1 c_1^{**}) = p_2 c_1^{**}$$

$$\therefore c_1^{**} = \frac{(1 + \delta)(1 + i)}{p_2 + p_1(1 + \delta)} = 0.988 \text{ (3 s.f.)}$$

$$c_2^{**} = (1 + i)^2 - (1 + i)c_1^{**} = 0.988 \text{ (3 s.f.)}$$

$$\therefore U(c_1^{**}, c_2^{**}) = \frac{p_1}{1 + \delta} \ln[c_1^{**}] + \frac{p_2}{(1 + \delta)^2} \ln[c_2^{**}] = -0.012 \text{ (3 s.f.)}$$

(c) Define

$$f(c) = \frac{p_1}{1 + \delta} \ln[c] + \frac{p_2}{(1 + \delta)^2} \ln[c] - (-0.512)$$
We wish to find \( c^* \) such that \( f(c^*) = 0 \). By interpolation we find \( c^* = 0.603 \). So the individual is indifferent between a strategy which uses the risk free asset only and one which achieves consumption of \((c_1, c_2) = (0.603, 0.603)\). This can be achieved with life insurance using wealth of only \( 1 \times \frac{0.603}{0.988} = 0.603 \). The individual would therefore be willing to forego up to \( \frac{0.988 - 0.603}{0.988} = 38.9\% \) of initial wealth of 1 for an actuarially fair market in life insurance. Equivalently, if life insurance was priced with loading less than \( \frac{0.988}{0.603} - 1 = 63.8\% \) then the individual could attain higher utility from purchasing life insurance, than from investing in the risk-free asset.

C.1. (a) Nya has a utility function of the form \( u(c) = -\exp(-\alpha (1 + r) w_0 + \alpha \tilde{y}) \). We wish to find \( \alpha \) to maximise \( \mathbb{E}[-\exp(-a((1 + r) w_0 + \alpha \tilde{y}))] \).

\[
H_N(\alpha) = \mathbb{E}[-\exp(-a((1 + r) w_0 + \alpha \tilde{y}))]
\]
\[
H'_N(\alpha) = \mathbb{E}[a \tilde{y} \exp(-a((1 + r) w_0 + \alpha \tilde{y}))]
\]
\[
H''_N(\alpha) = \mathbb{E}[-a^2 \tilde{y}^2 \exp(-a((1 + r) w_0 + \alpha \tilde{y}))] < 0
\]
\[
H'_N(\alpha^*_N = \frac{w_0}{2}) = 0
\]

Now we repeat for Lloyd with the presence of the background risk

\[
H_L(\alpha) = \mathbb{E}[-\exp(-a((1 + r) w_0 + \alpha \tilde{y} + \tilde{e}))]
\]
\[
H'_L(\alpha) = \mathbb{E}[a \tilde{y} \exp(-a((1 + r) w_0 + \alpha \tilde{y} + \tilde{e}))]
\]
\[
H''_L(\alpha) = \mathbb{E}[-a^2 \tilde{y}^2 \exp(-a((1 + r) w_0 + \alpha \tilde{y} + \tilde{e}))] < 0
\]
\[
H'_L(\alpha) = \int \int f_{\tilde{e}}(\tilde{e}) f_{\tilde{y}}(y) a y e^{-a((1 + r) w_0 + \alpha y + \tilde{e})} dy d\tilde{e}
\]
\[
= \int f_{\tilde{e}}(\tilde{e}) e^{-a \tilde{e}} \int f_{\tilde{y}}(y) a y e^{-a((1 + r) w_0 + \alpha y + \tilde{e})} dy d\tilde{e}
\]
\[
= \int f_{\tilde{e}}(\tilde{e}) e^{-a \tilde{e}} H_N'(\alpha) d\tilde{e}
\]

If we choose \( \alpha = \alpha^*_N \) then the internal integral will be zero, and so the overall integral is zero. We have shown that the solution is unique because \( H \) is concave in \( \alpha \), so this is unique solution. Lloyd will invest the same (absolute) amount in the risky asset as Nya, despite the presence of the background risk.

C.2. (a) \( \mathbb{E}[\tilde{x}] = 0 \times \frac{7}{10} + 4 \times \frac{1}{10} + 8 \times \frac{1}{10} + 10 \times \frac{1}{10} = 2.2 \)

(b) \( \mathbb{E}[\max(0, \tilde{x} - 3)] = 0 \times \frac{7}{10} + 1 \times \frac{1}{10} + 5 \times \frac{1}{10} + 7 \times \frac{1}{10} = 1.3 \)

\( \mathbb{E}[\max(0, \tilde{x} - 6)] = 0 \times \frac{7}{10} + 0 \times \frac{1}{10} + 2 \times \frac{1}{10} + 4 \times \frac{1}{10} = 0.6 \)

The actuarially fair premium is not proportional to the inverse of the deductible and so one would not expect a doubling of the deductible to halve the premium.
(c) Denoting the coinsurance rate \( \beta_d \) which gives the same premium as the premium with deductible \( d \), \( \beta_3 = \frac{13}{22} = 0.591 \) (3 s.f.) and \( \beta_6 = \frac{66}{122} = 0.273 \) (3 s.f.).

(d) For loss outcomes \((0, 4, 8, 10)\) net wealth on purchasing insurance with a deductible of 6 is \((9.4, 5.4, 3.4, 3.4)\) and net wealth on purchasing coinsurance with level of \( \beta_6 \) is \((9.40, 6.49, 3.58, 2.13)\). Denoting the cdf of final wealth if deductible is 6 by \( F \) and the cdf of final wealth if coinsurance is \( \beta_6 \) by \( G \) then the figure of cdfs is as follows:

Define \( S(w) := \int_0^w G(s) - F(s) \, ds \). Clearly \( S(2.13) = 0 \). Also \( S(9.4) = 0 \) since both products have the same mean wealth. Moreover \( G(s) - F(s) \geq 0 \) for \( s < 3.4 \) and \( G(s) - F(s) \leq 0 \) for \( s \geq 3.4 \) and so it must be that \( S(w) \) is weakly increasing from 0 from \( w = 2.13 \) to \( w = 3.4 \), and weakly decreasing to 0 from \( w = 3.4 \) to \( w = 9.4 \). By continuity of \( S \), \( S \) must always be nonnegative between 2.13 and 9.4.