S.4 Insurance and Saving

A.1. (a) Denoting the loss incurred by each member of a pool of size 2 and 3 by $\tilde{x}$ and $\tilde{y}$ respectively the pdf of these binomial random variables are:

$$\begin{aligned}
\mathbb{P}[\tilde{x} = 0] &= (1-p)^2 \\
\mathbb{P}[\tilde{x} = 100/2] &= 2p(1-p) \\
\mathbb{P}[\tilde{x} = 200/2] &= p^2 \\
\mathbb{P}[\tilde{y} = 0] &= (1-p)^3 \\
\mathbb{P}[\tilde{y} = 100/3] &= 3p(1-p)^2 \\
\mathbb{P}[\tilde{y} = 200/3] &= 3p^2(1-p) \\
\mathbb{P}[\tilde{y} = 300/3] &= p^3
\end{aligned}$$

(b) Let the distribution of $\tilde{\varepsilon}$ conditional on outcomes of $\tilde{y}$ be as follows

$$\begin{aligned}
(\tilde{\varepsilon}|\tilde{y} = 0) &= (0, 1) \\
(\tilde{\varepsilon}|\tilde{y} = 100/3) &= (-\frac{100}{3}, \frac{1}{3}; \frac{50}{3}, \frac{2}{3}) \\
(\tilde{\varepsilon}|\tilde{y} = 200/3) &= (-\frac{50}{3}, \frac{2}{3}; \frac{100}{3}, \frac{1}{3}) \\
(\tilde{\varepsilon}|\tilde{y} = 300/3) &= (0, 1)
\end{aligned}$$

Then $\mathbb{E}[\tilde{\varepsilon}|\tilde{y}] = 0$ for any outcome of $\tilde{y}$. Moreover, $\tilde{y} + \tilde{\varepsilon}$ has the same three possible outcomes as $\tilde{x}$ and these have the same probabilities:

$$\begin{aligned}
\mathbb{P}[\tilde{y} + \tilde{\varepsilon} = 0] &= \mathbb{P}[\tilde{y} = 0 \cap \tilde{\varepsilon} = 0] + \mathbb{P}[\tilde{y} = 100/3 \cap \tilde{\varepsilon} = -100/3] \\
&= (1-p)^3 \times 1 + 3p(1-p)^2 \times \frac{1}{3} \\
&= (1-p)^2 \times (1-p+p) = (1-p)^2 = \mathbb{P}[\tilde{x} = 0] \\
\mathbb{P}[\tilde{y} + \tilde{\varepsilon} = 50] &= \mathbb{P}[\tilde{y} = 100/3 \cap \tilde{\varepsilon} = 50/3] + \mathbb{P}[\tilde{y} = 200/3 \cap \tilde{\varepsilon} = -50/3] \\
&= 3p(1-p)^2 \times \frac{2}{3} + 3p^2(1-p) \times \frac{2}{3} \\
&= 2p(1-p)(1-p+p) = 2p(1-p) = \mathbb{P}[\tilde{x} = 50] \\
\mathbb{P}[\tilde{y} + \tilde{\varepsilon} = 100] &= \mathbb{P}[\tilde{y} = 100 \cap \tilde{\varepsilon} = 0] + \mathbb{P}[\tilde{y} = 200/3 \cap \tilde{\varepsilon} = 100/3] \\
&= p^3 \times 1 + 3p^2(1-p) \times \frac{1}{3} \\
&= p^2(p+1-p) = p^2 = \mathbb{P}[\tilde{x} = 100]
\end{aligned}$$

So any risk averse individual with starting wealth w, will prefer a pool with $N = 3$ to one with $N = 2$.

B.1. The agent's expected utility on wagering w is

$$\begin{aligned}
f(w) &= pu(X - w + 2w) + (1-p)u(X - w) \\
&= -pe^{-r(X+w)} - (1-p)e^{-r(X-w)}
\end{aligned}$$

where

$$\begin{aligned} f'(w) &= rpe^{-r(X+w)} - r(1-p)e^{-r(X-w)} \\ f''(w) &= -r^2pe^{-r(X+w)} - r^2(1-p)e^{-r(X-w)} < 0 \end{aligned}$$

Since f is twice differentiable with $f''(w) < 0$ for all w, the optimal choice of w, w^*, will solve $f'(w^*) = 0$.

$$\begin{aligned} rpe^{-r(X+w^*)} - r(1-p)e^{-r(X-w^*)} &= 0 \\ \therefore \frac{p}{1-p} &= e^{r(X+w^*)-r(X-w^*)} \\ \therefore w^* &= \frac{1}{2r}\ln\left(\frac{p}{1-p}\right) \end{aligned}$$

B.2. (a) Working in units of £1 million, the premium is $\frac{1}{4} \times 8 \times (1+0.2) = 2.4$.

(b) Denoting the expected utility on choosing coinsurance level β by $f(\beta)$, we have

$$\begin{aligned} f(\beta) &= \frac{3}{4}\ln(12-2.4\beta) + \frac{1}{4}\ln(12-2.4\beta-8(1-\beta)) \\ f'(\beta) &= -\frac{3}{4}\frac{2.4}{12-2.4\beta} + \frac{1}{4}\frac{5.6}{4+5.6\beta} \\ f''(\beta) &= -\frac{3}{4}\frac{2.4^2}{(12-2.4\beta)^2} - \frac{1}{4}\frac{5.6^2}{(4+5.6\beta)^2} < 0 \end{aligned}$$

So, f is twice differentiable and strictly concave, and so the optimal β, β^* solves $f'(\beta^*) = 0$ (or is at an endpoint $\beta^* = 0$ or $\beta^* = 1$):

$$\begin{aligned} -\frac{3}{4}\frac{2.4}{12-2.4\beta^*} + \frac{1}{4}\frac{5.6}{4+5.6\beta^*} &= 0 \\ \therefore 7.2(4+5.6\beta^*) &= 5.6(12-2.4\beta^*) \\ \therefore \beta^* &= \frac{5.6\times 12 - 7.2\times 4}{7.2\times 5.6 + 5.6\times 2.4} = \frac{38.4}{53.76} = 0.714 \text{ (3 s.f.)} \end{aligned}$$

Since this $\beta^* \in [0,1]$ and f is concave, the optimal solution will be β^*, not 0 or 1. (We will confirm this in the next part of the question.)

(c)

$$\begin{aligned} f(0) &= \frac{3}{4}\ln(12) + \frac{1}{4}\ln(4) = 2.210 \text{ (4 s.f.)} \\ f(\beta^*) &= \frac{3}{4}\ln\left(12 - 2.4\frac{38.4}{53.76}\right) + \frac{1}{4}\ln\left(4 + 5.6\frac{38.4}{53.76}\right) = 2.268 \text{ (4 s.f.)} \\ f(1) &= \ln(12-2.4) = 2.262 \text{ (4 s.f.)} \end{aligned}$$

(d) Full insurance ($\beta^* = 1$) is optimal if loading is zero (Mossin's Theorem).

B.3. For the standard portfolio problem let the utility function be u, satisfying $u' > 0$, $u'' < 0$, and initial wealth be w_0. Since the risk free (risky) investment earns return zero ($\tilde{x}$) an investment in the risky asset of α leads to realised wealth of $w_0 + \alpha\tilde{x}$ and expected utility, $f(\alpha)$, of:

$$f(\alpha) := \mathbb{E}\left[u(w_0 + \alpha\tilde{x})\right]$$

f is strictly concave and so the unique optimal α^* is the solution to the first order condition

$$f'(\alpha^*) = \mathbb{E}\left[\tilde{x}u'(w_0 + \alpha^*\tilde{x})\right] = 0.$$

(a) Now, let $\tilde{y} := (\tilde{x}, q; 0, 1-q)$, and let β denote the optimal amount of (new) asset $\tilde{y}$ to purchase. The new first order condition is

$$\mathbb{E}_{\tilde{y}}\left[\tilde{y}u'(w_0 + \beta^*\tilde{y})\right] = 0$$

Applying the law of total expectation ($\mathbb{E}_Y(Y) = \mathbb{E}_X(\mathbb{E}_Y(Y|X))$ we find

$$\begin{aligned}\mathbb{E}_{\tilde{y}}\left[\tilde{y}u'(w_0 + \beta^*\tilde{y})\right] &= \mathbb{E}_{\tilde{x}}\left[\mathbb{E}_{\tilde{y}}\left[\tilde{y}u'(w_0 + \beta^*\tilde{y})|\tilde{x}\right]\right] \\ &= \mathbb{E}_{\tilde{x}}\left[q\tilde{x}u'(w_0 + \beta^*\tilde{x}) + (1-q)\times 0\right] \\ &= q\mathbb{E}_{\tilde{x}}\left[\tilde{x}u'(w_0 + \beta^*\tilde{x})\right]\end{aligned}$$

Now the unique solution to this is $\beta^* = \alpha^*$, that is you purchase the same amount of the risky asset as you would have in the original problem. Note that this comes from the Independence Axiom: you prefer $w_0 + \alpha^*\tilde{x}$ to any other $w_0 + \alpha\tilde{x}$ and therefore prefer any stochastic mix of $w_0 + \alpha^*\tilde{x}$ and the risk-free asset to any stochastic mix of $w_0 + \alpha\tilde{x}$ and the risk-free asset, and so α^* is the optimal amount to purchase in both problems.

(b) Let $\tilde{z} := q\tilde{x}$, and let γ denote the optimal amount of (new) asset $\tilde{z}$ to purchase. The new first order condition is

$$\begin{aligned}\mathbb{E}_{\tilde{z}}\left[\tilde{z}u'(w_0 + \gamma^*\tilde{z})\right] &= 0 \\ \therefore \mathbb{E}_{\tilde{x}}\left[q\tilde{x}u'(w_0 + \gamma^*q\tilde{x})\right] &= 0\end{aligned}$$

From the definition of α^*, we have $\gamma^*q = \alpha^*$, and therefore $\gamma^* = \frac{\alpha^*}{q}$.

B.4. The question did not specify whether α can be negative, or whether the expected excess return from the risky asset was positive. We don't impose any restrictions in this solution.

(a) Let w_0 denote initial wealth and α denote the demand for the risky asset. The optimisation problem is

$$\max_{\alpha} p\ln(w_0(1+r) - \alpha(r-a)) + (1-p)\ln(w_0(1+r) + \alpha(b-r))$$

This is strictly concave in α and so the optimal choice of α, α^* satisfies first order condition

$$
\begin{aligned}
&-\frac{p(r-a)}{w_0(1+r)-\alpha^*(r-a)}+\frac{(1-p)(b-r)}{w_0(1+r)+\alpha^*(b-r)}=0\\
&\therefore p(r-a)(w_0(1+r)+\alpha^*(b-r))=(1-p)(b-r)(w_0(1+r)-\alpha^*(r-a))\\
&\therefore \alpha^*[p(r-a)(b-r)+(1-p)(r-a)(b-r)]=w_0(1+r)[(1-p)(b-r)-p(r-a)]\\
&\therefore \alpha^*=w_0(1+r)\left[\frac{1-p}{r-a}-\frac{p}{b-r}\right]\\
&\quad =w_0(1+r)\left[\frac{(1-p)(b-r)+p(a-r)}{(r-a)(b-r)}\right]
\end{aligned}
$$

(Note that α^* is the same sign as the excess expected return of the risky asset over the risk-free asset.)

(b) Yes, for $a < r < b$.

(c)

$$\frac{d\alpha^*}{dw_0}=\frac{\alpha^*}{w_0}$$

So a small increase in initial wealth increases the magnitude of α^*; if α^* is positive it becomes more positive, if α^* is negative it becomes more negative.

(d)

$$\frac{d\alpha^*}{dr}=\frac{\alpha^*}{1+r}-w_0(1+r)\left[\frac{1-p}{(r-a)^2}+\frac{p}{(b-r)^2}\right]$$

An increase in the risk free rate of return has two effects on the optimal choice of α. First there is a 'wealth' effect of $\frac{\alpha^*}{1+r}$, whereby by increasing the risk-free rate of return effectively increases the ex-post wealth of the individual, increasing the magnitude of α. Second by making the risky asset relatively less attractive it reduces α.

(e)

$$
\begin{aligned}
\frac{d\alpha^*}{da}&=w_0(1+r)\frac{1-p}{(r-a)^2}\geq 0\\
\frac{d\alpha^*}{db}&=w_0(1+r)\frac{p}{(b-r)^2}\geq 0
\end{aligned}
$$

An increase (decrease) in a or b leads to a first order stochastic dominant improvement (deterioration) in the returns from the risky asset, thereby increasing (decreasing) optimal demand.

(f) Let $c := (1-p)(b-r)+p(a-r)$ and so $r-a=\frac{(1-p)(b-r)-c}{p}$. Then

$$\alpha^*=w_0(1+r)\left[\frac{p}{b-r-\frac{c}{1-p}}-\frac{p}{b-r}\right]$$

Differentiating α^* with respect to b whilst holding expected excess return c constant gives

$$\frac{\partial \alpha^*}{\partial b} = w_0(1+r)p\left[-\frac{1}{(b-r-\frac{c}{1-p})^2}+\frac{1}{(b-r)^2}\right]$$

Since $b-r>0$ and $r-a>0$, and therefore $b-r-\frac{c}{1-p}>0$ we have $\frac{\partial \alpha^*}{\partial b}$ has the opposite sign to c, and therefore the opposite sign to α^*. So an increase in b whilst holding the expected excess return of the risky asset constant reduces the magnitude of α^*.

B.5. (a) The budget constraint is $1 = vc_1 + v^2c_2$ and so, substituting the budget constraint into the objective function, Sally's optimisation problem may be written as

$$\max_{c_1} \frac{p_1}{1+\delta}\ln[c_1] + \frac{p_2}{(1+\delta)^2}\ln[(1+i)^2-(1+i)c_1]$$

This is strictly concave in c_1 and so the optimal c_1 satisfies

$$\begin{aligned}
&\frac{p_1}{(1+\delta)c_1^*} - \frac{p_2}{(1+\delta)^2((1+i)-c_1^*)} = 0\\
&\therefore p_1(1+\delta)((1+i)-c_1^*) = p_2c_1^*\\
&\therefore c_1^* = \frac{p_1(1+\delta)(1+i)}{p_2+p_1(1+\delta)} = 0.741 \text{ (3 s.f.)}\\
&c_2^* = (1+i)^2-(1+i)c_1^* = 0.395 \text{ (3 s.f.)}\\
&\therefore U(c_1^*,c_2^*) = \frac{p_1}{1+\delta}\ln[c_1^*] + \frac{p_2}{(1+\delta)^2}\ln[c_2^*] = -0.512 \text{ (3 s.f.)}
\end{aligned}$$

(b) Now the budget constraint is $1 = p_1vc_1 + p_2v^2c_2$ and the optimisation problem is

$$\max_{c_1} \frac{p_1}{1+\delta}\ln[c_1] + \frac{p_2}{(1+\delta)^2}\ln\left[\frac{(1+i)^2-p_1(1+i)c_1}{p_2}\right]$$

which is also strictly concave and has first order condition

$$\begin{aligned}
&\frac{p_1}{(1+\delta)c_1^{**}} - \frac{p_2p_1}{(1+\delta)^2((1+i)-p_1c_1^{**})} = 0\\
&\therefore (1+\delta)((1+i)-p_1c_1^{**}) = p_2c_1^{**}\\
&\therefore c_1^{**} = \frac{(1+\delta)(1+i)}{p_2+p_1(1+\delta)} = 0.988 \text{ (3 s.f.)}\\
&c_2^{**} = \frac{(1+i)^2-(1+i)c_1^*}{p_2} = 0.988 \text{ (3 s.f.)}\\
&\therefore U(c_1^{**},c_2^{**}) = \frac{p_1}{1+\delta}\ln[c_1^{**}] + \frac{p_2}{(1+\delta)^2}\ln[c_2^{**}] = -0.012 \text{ (3 s.f.)}
\end{aligned}$$

(c) Define

$$f(c) = \frac{p_1}{1+\delta}\ln[c] + \frac{p_2}{(1+\delta)^2}\ln[c] - (-0.512)$$

We wish to find c^* such that $f(c^*) = 0$. By interpolation we find $c^* = 0.603$. So the individual is indifferent between a strategy which uses the risk free asset only and one which achieves consumption of $(c_1, c_2) = (0.603, 0.603)$. This can be achieved with life insurance using wealth of only $1 \times \frac{0.603}{0.988}$. The individual would therefore be willing to forego up to $\frac{0.988-0.603}{0.988} = 38.9\%$ of initial wealth of 1 for an actuarially fair market in life insurance. Equivalently, if life insurance was priced with loading less than $\frac{0.988}{0.603} - 1 = 63.8\%$ then the individual could attain higher utility from purchasing life insurance, than from investing in the risk-free asset.

C.1. (a) Nya has a utility function of the form $u(w) = -\exp(-aw)$ and so she chooses α to maximise $\mathbb{E}\left[-\exp\left(-a((1+r)w_0 + \alpha\tilde{y})\right)\right]$.

$$
\begin{aligned}
H_N(\alpha) &= \mathbb{E}\left[-\exp\left(-a((1+r)w_0 + \alpha\tilde{y})\right)\right] \\
H_N'(\alpha) &= \mathbb{E}\left[a\,\tilde{y}\exp\left(-a((1+r)w_0 + \alpha\tilde{y})\right)\right] \\
H_N''(\alpha) &= \mathbb{E}\left[-a^2\,\tilde{y}^2\exp\left(-a((1+r)w_0 + \alpha\tilde{y})\right)\right] < 0 \\
H_N'\left(\alpha_N^* = \frac{w_0}{2}\right) &= 0
\end{aligned}
$$

Now we repeat for Lloyd with the presence of the background risk

$$
\begin{aligned}
H_L(\alpha) &= \mathbb{E}\left[-\exp\left(-a((1+r)w_0 + \alpha\tilde{y} + \tilde{\varepsilon})\right)\right] \\
H_L'(\alpha) &= \mathbb{E}\left[a\,\tilde{y}\exp\left(-a((1+r)w_0 + \alpha\tilde{y} + \tilde{\varepsilon})\right)\right] \\
H_L''(\alpha) &= \mathbb{E}\left[-a^2\,\tilde{y}^2\exp\left(-a((1+r)w_0 + \alpha\tilde{y} + \tilde{\varepsilon})\right)\right] < 0 \\
H_L'(\alpha) &= \int\int f_{\tilde{\varepsilon}}(\varepsilon) f_{\tilde{y}}(y) a\,y e^{-a((1+r)w_0 + \alpha y + \varepsilon)} \mathrm{d}y\,\mathrm{d}\varepsilon \\
&= \int f_{\tilde{\varepsilon}}(\varepsilon) e^{-a\varepsilon} \int f_{\tilde{y}}(y) a\,y e^{-a((1+r)w_0 + \alpha y)} \mathrm{d}y\,\mathrm{d}\varepsilon \\
&= \int f_{\tilde{\varepsilon}}(\varepsilon) e^{-a\varepsilon}\, H_N'(\alpha)\,\mathrm{d}\varepsilon
\end{aligned}
$$

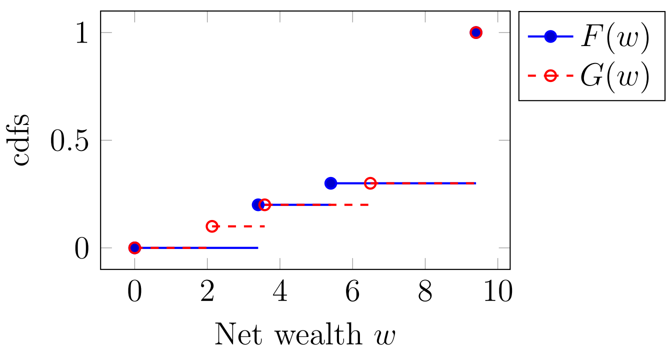
If we choose $\alpha = \alpha*_N$ then the internal integral will be zero, and so the overall integral is zero. We have shown that the solution is unique because H is concave in α, so this is unique solution. Lloyd will invest the same (absolute) amount in the risky asset as Nya, despite the presence of the background risk.

C.2. (a) $\mathbb{E}[\tilde{x}] = 0 \times \frac{7}{10} + 4 \times \frac{1}{10} + 8 \times \frac{1}{10} + 10 \times \frac{1}{10} = 2.2$

(b) $\mathbb{E}[\max(0, \tilde{x} - 3)] = 0 \times \frac{7}{10} + 1 \times \frac{1}{10} + 5 \times \frac{1}{10} + 7 \times \frac{1}{10} = 1.3$
$\mathbb{E}[\max(0, \tilde{x} - 6)] = 0 \times \frac{7}{10} + 0 \times \frac{1}{10} + 2 \times \frac{1}{10} + 4 \times \frac{1}{10} = 0.6$
The actuarially fair premium is not proportional to the inverse of the deductible and so one would not expect a doubling of the deductible to halve the premium.

(c) Denoting the coinsurance rate β_d which gives the same premium as the premium with deductible d, $\beta_3 = \frac{1.3}{2.2} = 0.591$ (3 s.f.) and $\beta_6 = \frac{0.6}{2.2} = 0.273$ (3 s.f.).

(d) For loss outcomes $(0, 4, 8, 10)$ net wealth on purchasing insurance with a deductible of 6 is $(9.4, 5.4, 3.4, 3.4)$ and net wealth on purchasing coinsurance with level of β_6 is $(9.40, 6.49, 3.58, 2.13)$. Denoting the cdf of final wealth if deductible is 6 by F and the cdf of final wealth if coinsurance is β_6 by G then the figure of cdfs is as follows:



Define $S(w) := \int_0^w G(s) - F(s)ds$. Clearly $S(2.13) = 0$. Also $S(9.4) = 0$ since both products have the same mean wealth. Moreover $G(s) - F(s) \geq 0$ for $s < 3.4$ and $G(s) - F(s) \leq 0$ for $s \geq 3.4$ and so it must be that $S(w)$ is weakly increasing from 0 from $w = 2.13$ to $w = 3.4$, and weakly decreasing to 0 from $w = 3.4$ to $w = 9.4$. By continuity of S, S must always be nonnegative between 2.13 and 9.4.