Part A Simulation Trinity 2013, Problem Sheet 3

1. Metropolis-Hastings algorithms.

   (a) Give a Metropolis-Hastings algorithm to sample according to the binomial probability mass function,

   $$\pi(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, 3, \dots, n$$

   with parameters n and p, where $0 < p < 1$ and $n \in \{1, 2, 3, \dots\}$. Use the proposal pmf $Y \sim U\{0, 1, 2, ..., n\}$ to get $X_t \to \text{Binomial}(n, p)$. Explain why the Markov chain is irreducible and aperiodic.

   (b) Give a Metropolis-Hastings algorithm to sample according to the Geometric probability mass function,

   $$\pi(x) = p(1-p)^{x-1}, \quad x = 1, 2, 3, \dots,$$

   with parameter p, where $0 < p < 1$. Use the proposal distribution $Y|X = x \sim U\{x-1, x+1\}$.

   (c) Give a Metropolis-Hastings algorithm to sample according to the Gamma probability density function,

   $$\pi(x) \propto x^{\alpha-1} \exp(-\beta x), \quad x > 0$$

   with parameters $\alpha, \beta > 0$. Use the proposal distribution $Y \sim \text{Exp}(\beta)$.

2. (The random-scan Gibbs sampler) Suppose $p(x)$ is the pmf of some multivariate random variable $X \in \mathbb{Z}^m$, so $x = (x_1, x_2, ..., x_m)$. Let

   $$x_{-i} = (x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_m)$$

   be the vector of components with x_i omitted. Consider a Metropolis-Hastings algorithm simulating a Markov chain $\{X^{(t)}\}_{t=0}^{\infty}$, with $X^{(t)} \to p$. Here is an update scheme: suppose $X^{(t)} = x$ and we propose a candidate y by choosing i at random from $1, 2, ..., m$, simulating y_i from the conditional distribution $X_i|X_{-i} = x_{-i}$ and setting

   $$y = (x_1, x_2, ..., x_{i-1}, y_i, x_{i+1}, ..., x_m).$$

   (a) Show that if y and x differ at a single index i then the pmf for $Y = y|X = x$ is

   $$q(y|x) = (1/m)p(y_i|x_{-i}).$$

   Give a formula for $p(y_i|x_{-i})$ in terms of $p(y)$.

   (b) Show that the acceptance probability $\alpha(y|x)$ is equal to one.

   (c) Write down the Metropolis Hastings algorithm for this case as simply as you can.

   (d) Give a Gibbs sampler for the multinomial probability mass function,

   $$\pi(y, z) = \frac{n!}{y!z!(n-y-z)!} p^y q^z (1-p-q)^{n-y-z}, \quad 0 \le y + z \le n$$

   with parameters n, p, q, where $p, q \ge 0$, $p + q \le 1$ and $n \in \{1, 2, 3, \dots\}$.

3. A *contingency table* X is an $n \times m$ matrix with non-negative integer entries $X_{i,j} \ge 0$ and fixed row sums $r = (r_1, r_2, ..., r_n)$ and fixed column sums $c = (c_1, c_2, ..., c_m)$. Denote by $\Omega_{r,c}$ the set of all such tables for given vectors r and c.

   An *index table* Y has row and column sums equal zero and entries $Y_{i,j} \in \{-1, 0, 1\}$. Denote by $U_{n,m}$ the uniform distribution on the set of all $n \times m$ index tables.

   Given $X \in \Omega_{r,c}$, the following algorithm generates a new random matrix X' with the same row and column sums as X.

Step 1 Simulate $Y \sim U_{n,m}$ [assume you have an algorithm simulating independent Y].

Step 2 Set $X' = X + Y$.

Here is an example.

$$X = \begin{bmatrix} 7 & 2 & 0 \\ 5 & 6 & 1 \\ 1 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad X' = X + Y = \begin{bmatrix} 7 & 2 & 0 \\ 5 & 7 & 0 \\ 1 & 5 & 4 \\ 0 & 1 & 2 \end{bmatrix}.$$

You may assume without proof that for any two contingency tables $A, B \in \Omega_{r,c}$ there is a finite sequence of index tables $Y_1, Y_2, ..., Y_K$ so that $B + \sum_{i=1}^{n} Y_i \in \Omega_{r,c}$ for $n = 1, 2, ..., K$ and $A = B + \sum_{i=1}^{K} Y_i$.

(a) Specify a Metropolis Hastings algorithm simulating a Markov chain, $X_t, t = 0, 1, 2, ...$ on $\Omega_{r,c}$ with equilibrium distribution π equal to the uniform distribution on $\Omega_{r,c}$.

(b) Show that the Markov chain in (a) is reversible with respect to π.

(c) Let $\{X_t\}_{t=0}^{\infty}$ be the Markov chain from part (a). Give sufficient conditions for $\{X_t\}_{t=0}^{\infty}$ to be ergodic in $\Omega_{r,c}$, and verify that these conditions are satisfied.