Part A Simulation Trinity 2013, Problem Sheet 1 (for classes in week 3)

1. (a) Let $Y \sim \text{Exp}(\lambda)$ and fix $a > 0$. Let $X = Y|Y \geq a$. That is, the random variable X is equal to Y conditioned on $Y \geq a$. Calculate $F_X(x)$ and $F_X^{-1}(u)$. Give an algorithm simulating X from $U \sim \mathcal{U}[0,1]$.

   (b) Let a and b be given, with $a < b$. Show that we can simulate $X = Y|a \leq Y \leq b$ from $U \sim \mathcal{U}[0,1]$ using
   $$X = F_Y^{-1}(F_Y(a)(1-U) + F_Y(b)U),$$
   i.e., show that if X is given by the formula above, then $\Pr(X \leq x) = \Pr(Y \leq x|a \leq Y \leq b)$. Apply the formula to simulate an exponential rv conditioned to be greater than a.

   (c) Here is a very simple algorithm simulating $X = Y|Y > a$ for $Y \sim \text{Exp}(\lambda)$:

   1 Let $Y \sim \text{Exp}(\lambda)$. Simulate $Y = y$.

   2 If $Y > a$ then stop and return $X = y$, and otherwise, start again at 1.

   Explain that this is just a rejection algorithm (give the proposal and target densities p and q and the bound $M = \max_x p(x)/q(x)$ and show the rejection algorithm reduces to the algorithm above). Calculate the expected number of trials to the first acceptance. Why is the inversion method to be preferred over rejection sampling for $a \gg 1/\lambda$?

2. Consider the family of distributions with probability density function (pdf)
   $$f_{\mu,\lambda}(x) = \lambda \exp(-2\lambda|x-\mu|), \qquad x \in \mathbb{R},$$
   where $\lambda > 0$ and $\mu \in \mathbb{R}$ are parameters.

   (a) Given $U \sim U(0,1)$, use the inversion method to simulate from $f_{\mu,\lambda}$.

   (b) Let X have pdf $f_{\mu,\lambda}$. Show that $a + bX$ has pdf $f_{\mu',\lambda'}$ for $b \neq 0$. Find the parameters μ', λ'.

   (c) Let $Y, Z \sim \text{Exp}(r)$. Show that $Y - Z$ has pdf $f_{\mu',\lambda'}$. Find the parameters μ', λ'. Hence, use the transformation method to simulate from $f_{\mu,\lambda}$ for any $\lambda > 0$ and $\mu \in \mathbb{R}$, given $U_1, U_2 \sim U(0,1)$ independent.

   (d) Implement an R function to simulate from $f_{\mu,\lambda}$ using the algorithm derived in part (c). For $\mu = 5$, $\lambda = 2$, simulate 1000 draws using your function and plot a histogram of the sample.

3. Suppose X is a discrete rv taking values $X \in \{1, 2, ..., m\}$ with pmf $p(i) = \Pr(X = i)$. Let $q(i) = 1/m$ be the pmf of the uniform distribution. Give a rejection algorithm simulating $X \sim p$ using proposals Y distributed according to q. Calculate the expected number of simulations $Y \sim q$ per returned value of X if $p = (0.5, 0.25, 0.125, 0.125)$.

4. Let $Y \sim q$ with density $q(x) \propto \exp(-|x|)$ for $x \in \mathbb{R}$. Let $X \sim \text{N}(0,1)$ be a standard normal random variable, density $p(x) \propto \exp(-x^2/2)$.

   (a) Find M to bound $p(x)/q(x)$ for all real x.

   (b) Give a rejection algorithm simulating X using $Y \sim q$ as the proposal distribution.

   (c) Can we simulate $Y \sim q$ by rejection using $X \sim p$ to give the proposal distribution?