An Introduction to Bayesian Nonparametric Modelling

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Outline

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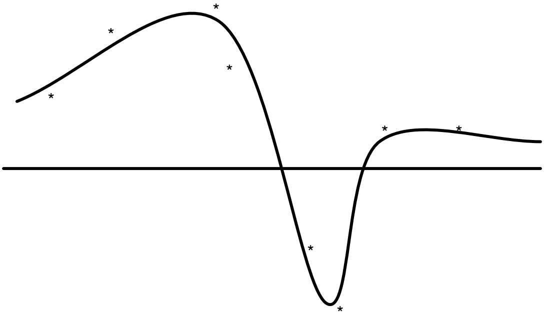
Regression with Basis Functions

- Supervised learning of a function $f^*: \mathbb{X} \to \mathbb{Y}$ from training data $\{x_i, y_i\}_{i=1}^n$.



Regression with Basis Functions

- Assume a set of basis functions $\phi_1, \ldots, \phi_K$ and parametrize a function:

$$f(x; \mathbf{w}) = \sum_{k=1}^{K} w_k \phi_k(x)$$

  Parameters $\mathbf{w} = \{w_1, \ldots, w_K\}$.

- Find optimal parameters

$$\underset{\mathbf{w}}{\text{argmin}} \sum_{i=1}^{n} \left| y_i - f(x_i; \mathbf{w}) \right|^2 = \underset{\mathbf{w}}{\text{argmin}} \sum_{i=1}^{n} \left| y_i - \textstyle\sum_{k=1}^{K} w_k \phi_k(x_i) \right|^2$$

- We will be Bayesian in this lecture, so we need to rephrase using probabilistic model with priors on parameters:

$$y_i | x_i, \mathbf{w} = f(x_i; \mathbf{w}) + \epsilon_i \qquad \epsilon_i \sim \mathcal{N}(0, \sigma^2)$$
$$w_k \sim \mathcal{N}(0, \tau^2)$$

- Computer posterior $p(\mathbf{w} | \{x_i, y_i\})$.

Regression with Basis Functions

$$f(x; \mathbf{w}) = \sum_{k=1}^{K} w_k \phi_k(x)$$

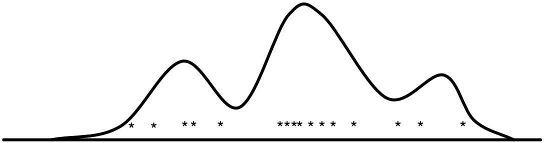
- What basis functions to use?
- How many basis functions to use?
- Do we really believe that the true $f^*(x)$ can be expressed as $f^*(x) = f(x; \mathbf{w}^*)$ for some $\mathbf{w}^*$?

$$\epsilon_i \sim \mathcal{N}(0, \sigma^2)$$

- Do we believe the noise process is Gaussian?

Density Estimation with Mixture Models

- Unsupervised learning of a density $f^*(x)$ from training samples $\{x_i\}$.



- Perhaps use an exponential family distribution, e.g. Gaussian?

$$\mathcal{N}(x; \mu, \Sigma) = |2\pi\Sigma|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)\right)$$

  Unimodal, restrictive shape, light tail...

- Use a mixture model instead,

$$f(x) = \sum_{k=1}^{K} \pi_k \mathcal{N}(x; \mu_k, \Sigma_k)$$

- Do we believe that the true density is a mixture of K components?
- How many mixture components to use?

Latent Variable Modelling

- Say we have n vector observations $x_1, \ldots, x_n$.
- Model each observation as a linear combination of K latent sources:

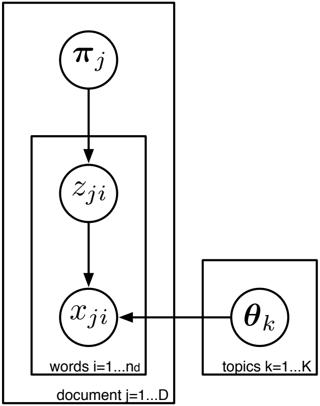
$$x_i = \sum_{k=1}^{K} \Lambda_k y_{ik} + \epsilon_i$$

  y_{ik}: activity of source k in datum i.
  Λ_k: basis vector describing effect of source k.
- Examples include principle components analysis, factor analysis, independent components analysis.
- How many sources are there?
- Do we believe that K sources is sufficient to explain all our data?
- What prior distribution should we use for sources?

Topic Modelling with Latent Dirichlet Allocation

- Infer topics from a document corpus, topics being sets of words that tend to co-occur together.
- Using (Bayesian) latent Dirichlet allocation:

$$\begin{aligned}
\boldsymbol{\pi}_j &\sim \text{Dirichlet}(\tfrac{\alpha}{K}, \ldots, \tfrac{\alpha}{K}) \\
\boldsymbol{\theta}_k &\sim \text{Dirichlet}(\tfrac{\beta}{W}, \ldots, \tfrac{\beta}{W}) \\
z_{ji} | \boldsymbol{\pi}_j &\sim \text{Multinomial}(\boldsymbol{\pi}_j) \\
x_{ji} | z_{ji}, \boldsymbol{\theta}_{z_{ji}} &\sim \text{Multinomial}(\boldsymbol{\theta}_{z_{ji}})
\end{aligned}$$



- How many topics can we find from the corpus?

Outline

Modelling Data

- Models are almost never correct for real world data.
- How do we deal with model misfit?
  - Quantify closeness to true model, and optimality of fitted model;
  - Model selection or averaging;
  - Increase the flexibility of your model class.
- Bayesian nonparametrics are good solutions from the second and third perspectives.

Model Selection and Model Averaging

- Data $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$.
- Model M_k parametrized by θ_k, for $k = 1, 2, \ldots$.
- Marginal likelihood:

$$p(\mathbf{x}|M_k) = \int p(\mathbf{x}|\theta_k, M_k) p(\theta_k, M_k) d\theta_k$$

- Model selection and averaging:

$$M = \underset{M_k}{\operatorname{argmax}}\, p(\mathbf{x}|M_k) \quad \text{or} \quad p(k, \theta_k|\mathbf{x}) = \frac{p(k)p(\theta_k|M_k)p(\mathbf{x}|\theta_k, M_k)}{\sum_{k'} p(k')p(\theta_{k'}|M_{k'})p(\mathbf{x}|\theta_{k'}, M_{k'})}$$

- Model selection and averaging is to prevent overfitting and underfitting, and are usually expense to compute.
- But reasonable and proper Bayesian methods should not overfit anyway [Rasmussen and Ghahramani 2001].

Nonparametric Modelling

- What is a nonparametric model?
  - A really large parametric model;
  - A parametric model where the number of parameters increases with data;
  - A model over infinite dimensional function or measure spaces.
  - A family of distributions that is dense in some large space.
- Why nonparametric models in Bayesian theory of learning?
  - broad class of priors that allows data to "speak for itself";
  - side-step model selection and averaging.
- How do we deal with the very large parameter spaces?
  - Marginalize out all but a finite number of parameters;
  - Define infinite space implicitly (akin to the kernel trick) using either Kolmogorov Consistency Theorem or de Finetti's Theorem.

Gaussian Processes

- A *Gaussian process* (GP) is a random function $f : \mathbb{X} \to \mathbb{R}$ such that for any finite set of input points $x_1, \dots, x_n$,

$$\begin{bmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} m(x_1) \\ \vdots \\ m(x_n) \end{bmatrix}, \begin{bmatrix} c(x_1, x_1) & \dots & c(x_1, x_n) \\ \vdots & \ddots & \vdots \\ c(x_n, x_1) & \dots & c(x_n, x_n) \end{bmatrix} \right)$$

  where the parameters are the mean function $m(x)$ and covariance kernel $c(x, y)$.
- Note: a random function f is a stochastic process. It is a collection of random variables $\{f(x)\}_{x \in \mathbb{X}}$ one for each possible input value x.
- Can also be expressed as

$$f(x) = \sum_{k=1}^{K} w_k \phi_k(x) \quad \text{as } K \to \infty.$$

[Rasmussen and Williams 2006]

Posterior and Predictive Distributions

- How do we compute the posterior and predictive distributions?
- Training set $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ and test input x_{n+1}.
- Out of the (uncountably infinitely) many random variables $\{f(x)\}_{x \in \mathbb{X}}$ making up the GP only $n+1$ has to do with the data:

$$f(x_1), f(x_2), \ldots, f(x_{n+1})$$

- Training data gives observations $f(x_1) = y_1, \ldots, f(x_n) = y_n$. The predictive distribution of $f(x_{n+1})$ is simply

$$p(f(x_{n+1}) | f(x_1) = y_1, \ldots, f(x_n) = y_n)$$

  which is easy to compute since $f(x_1), \ldots, f(x_{n+1})$ is Gaussian.
- This can be generalized to noisy observations $y_i = f(x_i) + \epsilon_i$ or non-linear effects $y_i \sim D(f(x_i))$ where $D(\theta)$ is a distribution parametrized by θ.

Consistency and Existence

- The definition of Gaussian processes only give finite dimensional marginal distributions of the stochastic process.
- Fortunately these marginal distributions are *consistent*.
  - For every finite set $\mathbf{x} \subset \mathbb{X}$ we have a distinct distribution $p_{\mathbf{x}}([f(x)]_{x \in \mathbf{x}})$. These distributions are said to be consistent if

$$p_{\mathbf{x}}([f(x)]_{x \in \mathbf{x}}) = \int p_{\mathbf{x} \cup \mathbf{y}}([f(x)]_{x \in \mathbf{x} \cup \mathbf{y}}) d[f(x)]_{x \in \mathbf{y}}$$

    for disjoint and finite $\mathbf{x}, \mathbf{y} \subset \mathbb{X}$.
  - The marginal distributions for the GP are consistent because *Gaussians are closed under marginalization*.
- The *Kolmogorov Consistency Theorem* guarantees existence of GPs, i.e. the whole stochastic process $\{f(x)\}_{x \in \mathbb{X}}$.
  - Further information in Peter Orbanz' Bayesian nonparametric tutorial.

Outline

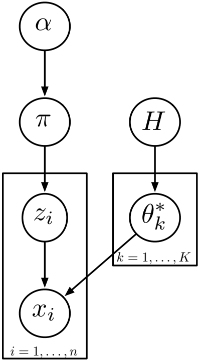
Bayesian Mixture Models

- Let's be Bayesian about mixture models, and place priors over our parameters (and to compute posteriors).
- First, introduce variable z_i indicator which component x_i belongs to.

$$z_i | \boldsymbol{\pi} \sim \text{Multinomial}(\boldsymbol{\pi})$$
$$x_i | z_i = k, \boldsymbol{\mu}, \boldsymbol{\Sigma} \sim \mathcal{N}(\mu_k, \Sigma_k)$$

- Second, introduce conjugate priors for parameters:

$$\boldsymbol{\pi} \sim \text{Dirichlet}(\tfrac{\alpha}{K}, \ldots, \tfrac{\alpha}{K})$$
$$\mu_k, \Sigma_k = \theta_k^* \sim H = \mathcal{N}\text{-}\mathcal{IW}(0, s, d, \Phi)$$

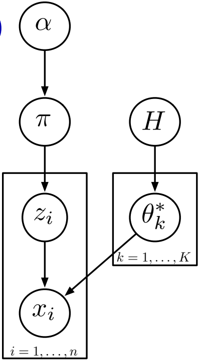


[Rasmussen 2000]

Gibbs Sampling for Bayesian Mixture Models

- All conditional distributions are simple to compute:

$$\begin{aligned} p(z_i = k|\text{others}) &\propto \pi_k \mathcal{N}(x_i; \mu_k, \Sigma_k) \\ \boldsymbol{\pi}|\mathbf{z} &\sim \text{Dirichlet}(\tfrac{\alpha}{K} + n_1(\mathbf{z}), \ldots, \tfrac{\alpha}{K} + n_K(\mathbf{z})) \\ \mu_k, \Sigma_k|\text{others} &\sim \mathcal{N}\text{-}\mathcal{IW}(\nu', s', d', \Phi') \end{aligned}$$



- Not as efficient as collapsed Gibbs sampling which integrates out $\boldsymbol{\pi}, \boldsymbol{\mu}, \boldsymbol{\Sigma}$:

$$p(z_i = k|\text{others}) \propto \frac{\frac{\alpha}{K} + n_k(\mathbf{z}_{-i})}{\alpha + n - 1} \times p(x_i|\{x_{i'} : i' \neq i, z_{i'} = k\})$$

- Demo: fm_demointeractive.

Infinite Bayesian Mixture Models

- We will take $K \to \infty$.
- Imagine a very large value of K.
- There are at most $n < K$ occupied components, so most components are *empty*. We can lump these empty components together:

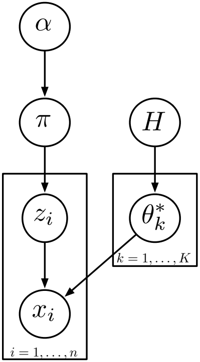
  Occupied clusters:

  $$p(z_i = k|\text{others}) \propto \frac{\frac{\alpha}{K} + n_k(\mathbf{z}_{-i})}{n - 1 + \alpha} p(x_i|\mathbf{x}_k^{-i})$$

  Empty clusters:

  $$p(z_i = k_{\text{empty}}|\mathbf{z}^{-i}) \propto \frac{\alpha \frac{K - K^*}{K}}{n - 1 + \alpha} p(x_i|\{\})$$

- Demo: dpm_demointeractive.



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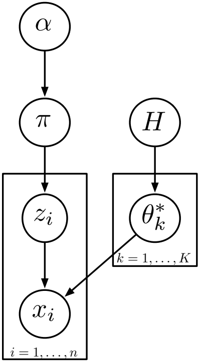


Infinite Bayesian Mixture Models

- The actual infinite limit of finite mixture models does not make sense: any particular component will get a mixing proportion of 0.
- In the Gibbs sampler we bypassed this by lumping empty clusters together.
- Other better ways of making this infinite limit precise:
  - Look at the prior clustering structure induced by the Dirichlet prior over mixing proportions—*Chinese restaurant process*.
  - Re-order components so that those with larger mixing proportions tend to occur first, before taking the infinite limit—*stick-breaking construction*.
- Both are different views of the *Dirichlet process* (DP).
- DPs can be thought of as infinite dimensional Dirichlet distributions.
- The $K \to \infty$ Gibbs sampler is for DP mixture models.

Outline

A Tiny Bit of Measure Theoretic Probability Theory

- A σ-*algebra* Σ is a family of subsets of a set Θ such that
  - Σ is not empty;
  - If $A \in \Sigma$ then $\Theta \backslash A \in \Sigma$;
  - If $A_1, A_2, \ldots \in \Sigma$ then $\cup_{i=1}^{\infty} A_i \in \Sigma$.
- (Θ, Σ) is a *measure space* and $A \in \Sigma$ are the *measurable sets*.
- A *measure* μ over (Θ, Σ) is a function $\mu : \Sigma \to [0, \infty]$ such that
  - $\mu(\emptyset) = 0$;
  - If $A_1, A_2, \ldots \in \Sigma$ are disjoint then $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.
  - Everything we consider here will be measurable.
  - A probability measure is one where $\mu(\Theta) = 1$.
- Given two measure spaces (Θ, Σ) and (Δ, Φ), a function $f : \Theta \to \Delta$ is *measurable* if $f^{-1}(A) \in \Sigma$ for every $A \in \Phi$.

A Tiny Bit of Measure Theoretic Probability Theory

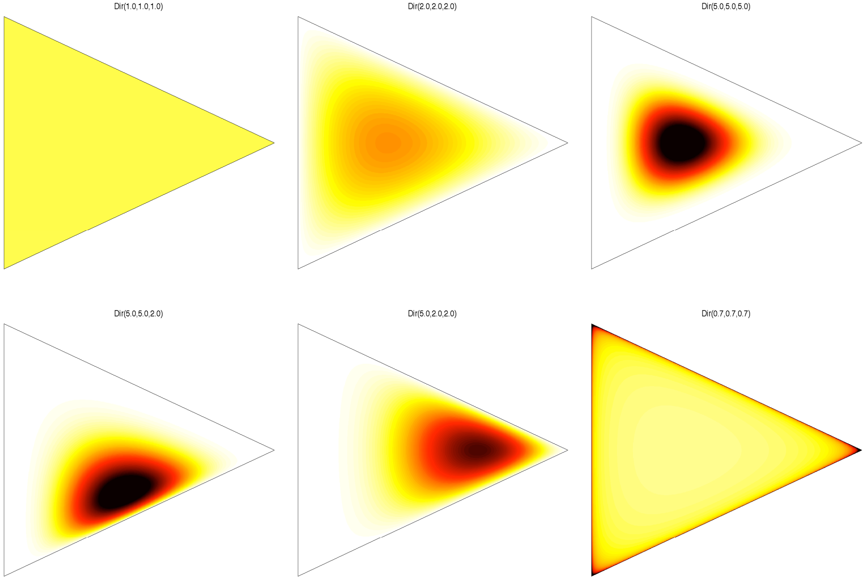
- If p is a probability measure on (Θ, Σ), a *random variable* X taking values in Δ is simply a measurable function $X : \Theta \to \Delta$.
  - Think of the probability space (Θ, Σ, p) as a black-box random number generator, and X as a function taking random samples in Θ and producing random samples in Δ.
  - The probability of an event $A \in \Phi$ is $p(X \in A) = p(X^{-1}(A))$.
- A *stochastic process* is simply a collection of random variables $\{X_i\}_{i \in \mathbb{I}}$ over the same measure space (Θ, Σ), where $\mathbb{I}$ is an index set.
  - What distinguishes a stochastic process from, say, a graphical model is that $\mathbb{I}$ can be infinite, even uncountably so.
  - This raises issues of how do you even define them and how do you ensure that they can even existence (mathematically speaking).
- Stochastic processes form the core of many Bayesian nonparametric models.
  - Gaussian processes, Poisson processes, gamma processes, Dirichlet processes, beta processes...

Outline

Dirichlet Distributions

- A *Dirichlet distribution* is a distribution over the K-dimensional probability simplex:
$$\Delta_K = \left\{(\pi_1, \ldots, \pi_K) \,:\, \pi_k \geq 0, \sum_k \pi_k = 1\right\}$$
- We say $(\pi_1, \ldots, \pi_K)$ is Dirichlet distributed,
$$(\pi_1, \ldots, \pi_K) \sim \text{Dirichlet}(\lambda_1, \ldots, \lambda_K)$$
with parameters $(\lambda_1, \ldots, \lambda_K)$, if
$$p(\pi_1, \ldots, \pi_K) = \frac{\Gamma(\sum_k \lambda_k)}{\prod_k \Gamma(\lambda_k)} \prod_{k=1}^{n} \pi_k^{\lambda_k - 1}$$
- Equivalent to normalizing a set of independent gamma variables:
$$(\pi_1, \ldots, \pi_K) = \tfrac{1}{\sum_k \gamma_k}(\gamma_1, \ldots, \gamma_K)$$
$$\gamma_k \sim \text{Gamma}(\lambda_k) \qquad \text{for } k = 1, \ldots, K$$

Dirichlet Distributions

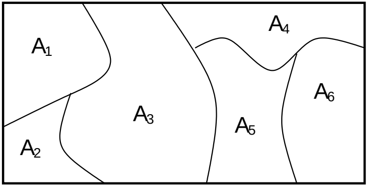


Dirichlet Processes

- A *Dirichlet Process* (DP) is a random probability measure G over (Θ, Σ) such that for any finite set of measurable partitions $A_1 \dot\cup \ldots \dot\cup A_K = \Theta$,

$$(G(A_1), \ldots, G(A_K)) \sim \text{Dirichlet}(\lambda(A_1), \ldots, \lambda(A_K))$$

where λ is a base measure.



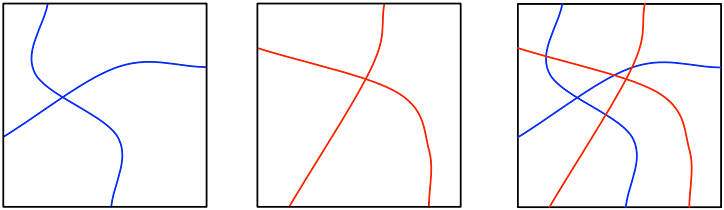
- The above family of distributions is consistent (next slide), and *Kolmogorov Consistency Theorem* can be applied to show existence (but there are technical conditions restricting the generality of the definition).

[Ferguson 1973, Blackwell and MacQueen 1973]

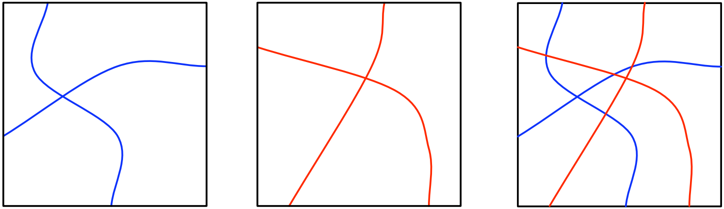
Consistency of Dirichlet Marginals

- If we have two partitions $(A_1, \ldots, A_K)$ and $(B_1, \ldots, B_J)$ of Θ, how do we see if the two Dirichlets are consistent?
- Because Dirichlet variables are normalized gamma variables and sums of gammas are gammas, if $(I_1, \ldots, I_j)$ is a partition of $(1, \ldots, K)$,

$$\left(\sum_{i \in I_1} \pi_i, \ldots, \sum_{i \in I_j} \pi_i\right) \sim \text{Dirichlet}\left(\sum_{i \in I_1} \lambda_i, \ldots, \sum_{i \in I_j} \lambda_i\right)$$



Consistency of Dirichlet Marginals



- Form the common refinement $(C_1, \ldots, C_L)$ where each C_ℓ is the intersection of some A_k with some B_j. Then:

$$\begin{aligned}
\text{By definition, } (G(C_1), \ldots, G(C_L)) &\sim \text{Dirichlet}(\lambda(C_1), \ldots, \lambda(C_L)) \\
(G(A_1), \ldots, G(A_K)) &= \left(\textstyle\sum_{C_\ell \subset A_1} G(C_\ell), \ldots, \sum_{C_\ell \subset A_K} G(C_\ell)\right) \\
&\sim \text{Dirichlet}(\lambda(A_1), \ldots, \lambda(A_K)) \\
\text{Similarly, } (G(B_1), \ldots, G(B_J)) &\sim \text{Dirichlet}(\lambda(B_1), \ldots, \lambda(B_J))
\end{aligned}$$

  so the distributions of $(G(A_1), \ldots, G(A_K))$ and $(G(B_1), \ldots, G(B_J))$ are consistent.

- Demonstration: DPgenerate.

Parameters of Dirichlet Processes

- Usually we split the λ base measure into two parameters $\lambda = \alpha H$:
  - *Base distribution* H, which is like the *mean* of the DP.
  - *Strength parameter* α, which is like an *inverse-variance* of the DP.
- We write:

$$G \sim \mathrm{DP}(\alpha, H)$$

  if for any partition $(A_1, \ldots, A_K)$ of Θ:

$$(G(A_1), \ldots, G(A_K)) \sim \mathrm{Dirichlet}(\alpha H(A_1), \ldots, \alpha H(A_K))$$

- The first and second moments of the DP:

$$\text{Expectation:} \quad \mathbb{E}[G(A)] = H(A)$$

$$\text{Variance:} \quad \mathbb{V}[G(A)] = \frac{H(A)(1 - H(A))}{\alpha + 1}$$

  where A is any measurable subset of Θ.

Representations of Dirichlet Processes

- Draws from Dirichlet processes will always place all their mass on a countable set of points:

$$G = \sum_{k=1}^{\infty} \pi_k \delta_{\theta_k^*}$$

  where $\sum_k \pi_k = 1$ and $\theta_k^* \in \Theta$.

  - What is the joint distribution over $\pi_1, \pi_2, \ldots$ and $\theta_1^*, \theta_2^*, \ldots$?

- Since G is a (random) probability measure over Θ, we can treat it as a distribution and draw samples from it. Let

$$\theta_1, \theta_2, \ldots \sim G$$

  be random variables with distribution G.

  - What is the marginal distribution of $\theta_1, \theta_2, \ldots$ with G integrated out?
  - There is positive probability that sets of θ_i's can take on the same value θ_k^* for some k, i.e. the θ_i's cluster together. How do these clusters look like?
  - For practical modelling purposes this is sufficient. But is this sufficient to tell us all about G?

Stick-breaking Construction

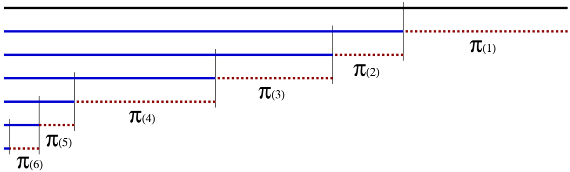
$$G = \sum_{k=1}^{\infty} \pi_k \delta_{\theta_k^*}$$

- There is a simple construction giving the joint distribution of $\pi_1, \pi_2, \ldots$ and $\theta_1^*, \theta_2^*, \ldots$ called the *stick-breaking construction*.

$$\theta_k^* \sim H$$

$$\pi_k = v_k \prod_{i=1}^{k-1} (1 - v_i)$$

$$v_k \sim \mathrm{Beta}(1, \alpha)$$



- Also known as the *GEM* distribution, write $\boldsymbol{\pi} \sim \mathrm{GEM}(\alpha)$.

[Sethuraman 1994]

Posterior of Dirichlet Processes

- Since G is a probability measure, we can draw samples from it,

$$\begin{aligned} G &\sim \mathrm{DP}(\alpha, H) \\ \theta_1, \ldots, \theta_n | G &\sim G \end{aligned}$$

  What is the posterior of G given observations of $\theta_1, \ldots, \theta_n$?

- The usual Dirichlet-multinomial conjugacy carries over to the nonparametric DP as well:

$$G | \theta_1, \ldots, \theta_n \sim \mathrm{DP}(\alpha + n, \tfrac{\alpha H + \sum_{i=1}^n \delta_{\theta_i}}{\alpha + n})$$

Pólya Urn Scheme

$$\theta_1, \theta_2, \ldots \sim G$$

- The marginal distribution of $\theta_1, \theta_2, \ldots$ has a simple generative process called the *Pólya urn scheme*.

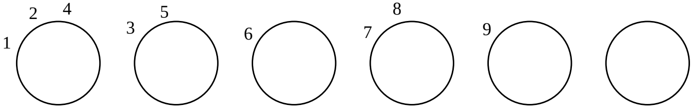
$$\theta_n | \theta_{1:n-1} \sim \frac{\alpha H + \sum_{i=1}^{n-1} \delta_{\theta_i}}{\alpha + n - 1}$$

- Picking balls of different colors from an urn:
  - Start with no balls in the urn.
  - with probability $\propto \alpha$, draw $\theta_n \sim H$, and add a ball of color θ_n into urn.
  - With probability $\propto n - 1$, pick a ball at random from the urn, record θ_n to be its color and return two balls of color θ_n into urn.
- Pólya urn scheme is like a “representer” for the DP—a finite projection of an infinite object G.
- Also known as the *Blackwell-MacQueen urn scheme*.

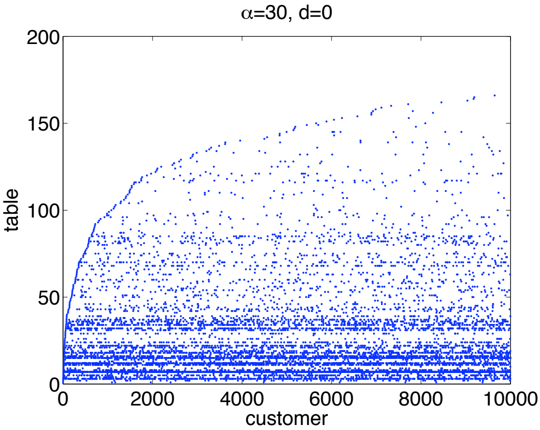
[Blackwell and MacQueen 1973]

Chinese Restaurant Process

- $\theta_1, \ldots, \theta_n$ take on $K < n$ distinct values, say $\theta_1^*, \ldots, \theta_K^*$.
- This defines a partition of $(1, \ldots, n)$ into K clusters, such that if i is in cluster k, then $\theta_i = \theta_k^*$.
- The distribution over partitions is a *Chinese restaurant process* (CRP).
- Generating from the CRP:
  - First customer sits at the first table.
  - Customer n sits at:
    - Table k with probability $\frac{n_k}{\alpha+n-1}$ where n_k is the number of customers at table k.
    - A new table $K+1$ with probability $\frac{\alpha}{\alpha+n-1}$.
  - Customers $\Leftrightarrow$ integers, tables $\Leftrightarrow$ clusters.



Chinese Restaurant Process



- The CRP exhibits the *clustering property* of the DP.
  - *Rich-gets-richer* effect implies small number of large clusters.
  - Expected number of clusters is $K = O(\alpha \log n)$.

Exchangeability

- Instead of deriving the Pólya urn scheme by marginalizing out a DP, consider starting directly from the conditional distributions:

$$\theta_n | \theta_{1:n-1} \sim \frac{\alpha H + \sum_{i=1}^{n-1} \delta_{\theta_i}}{\alpha + n - 1}$$

- For any n, the joint distribution of $\theta_1, \dots, \theta_n$ is:

$$p(\theta_1, \dots, \theta_n) = \frac{\alpha^K \prod_{k=1}^{K} h(\theta_k^*)(m_{nk} - 1)!}{\prod_{i=1}^{n} i - 1 + \alpha}$$

  where $h(\theta)$ is density of θ under H, $\theta_1^*, \dots, \theta_K^*$ are the unique values, and θ_k^* occurred m_{nk} times among $\theta_1, \dots, \theta_n$.

- The joint distribution is *exchangeable* wrt permutations of $\theta_1, \dots, \theta_n$.
- *De Finetti's Theorem* says that there must be a random probability measure G making $\theta_1, \theta_2, \dots$ iid. This is the DP.

De Finetti's Theorem

Let $\theta_1, \theta_2, \ldots$ be an infinite sequence of random variables with joint distribution p. If for all $n \geq 1$, and all permutations $\sigma \in \Sigma_n$ on n objects,

$$p(\theta_1, \ldots, \theta_n) = p(\theta_{\sigma(1)}, \ldots, \theta_{\sigma(n)})$$

That is, the sequence is infinitely exchangeable. Then there exists a (unique) latent random parameter G such that:

$$p(\theta_1, \ldots, \theta_n) = \int p(G) \prod_{i=1}^{n} p(\theta_i | G) dG$$

where ρ is a joint distribution over G and θ_i's.

- θ_i's are *independent* given G.
- Sufficient to define G through the conditionals $p(\theta_n | \theta_1, \ldots, \theta_{n-1})$.
- G can be *infinite dimensional* (indeed it is often a *random measure*).
- The set of infinitely exchangeable sequences is convex and it is an important theoretical topic to study the set of extremal points.
- Partial exchangeability: Markov, group, arrays,...

Outline

Binary Latent Variable Models

- Consider a latent variable model with binary sources/features,

$$z_{ik} = \begin{cases} 1 & \text{with probability } \mu_k; \\ 0 & \text{with probability } 1 - \mu_k. \end{cases}$$

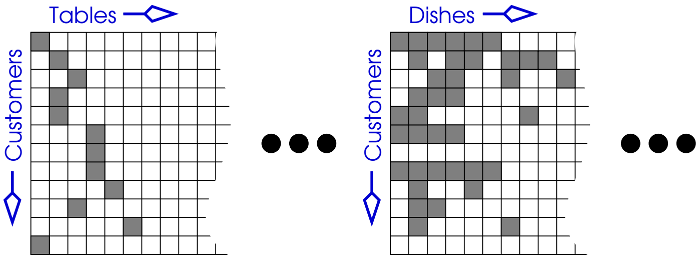
- Example: Data items could be movies like “Terminator 2”, “Shrek” and “Lord of the Rings”, and features could be “science fiction”, “fantasy”, “action” and “Arnold Schwarzenegger”.
- Place beta prior over the probabilities of features:

$$\mu_k \sim \text{Beta}(\tfrac{\alpha}{K}, 1)$$

- We will again take $K \to \infty$.

Indian Buffet Processes

- The *Indian Buffet Process* (IBP) is akin to the Chinese restaurant process but describes each customer with a binary vector instead of cluster.
- Generating from an IBP:
  - Parameter α.
  - First customer picks Poisson(α) dishes to eat.
  - Subsequent customer i picks dish k with probability $\frac{m_k}{i}$; and picks Poisson($\frac{\alpha}{i}$) new dishes.



Indian Buffet Processes and Exchangeability

- The IBP is infinitely exchangeable. For this to make sense, we need to "forget" the ordering of the dishes.
  - "Name" each dish k with a Λ_k^* drawn iid from H.
  - Each customer now eats a set of dishes: $\Psi_i = \{\Lambda_k^* : z_{ik} = 1\}$.
  - The joint probability of $\Psi_1, \ldots, \Psi_n$ can be calculated:

$$p(\Psi_1, \ldots, \Psi_n) = \exp\left(-\alpha \sum_{i=1}^{n} \frac{1}{i}\right) \alpha^K \prod_{k=1}^{K} \frac{(m_k - 1)!(n - m_k)!}{n!} h(\Lambda_k^*)$$

  K: total number of dishes tried by n customers.
  Λ_k^*: Name of kth dish tried.
  m_k: number of customers who tried dish Λ_k^*.

- De Finetti's Theorem again states that there is some random measure underlying the IBP.
- This random measure is the beta process.

[Griffiths and Ghahramani 2006, Thibaux and Jordan 2007]

Beta Processes

- A *beta process* $B \sim \mathrm{BP}(c, \alpha H)$ is a random discrete measure with form:

$$B = \sum_{k=1}^{\infty} \mu_k \delta_{\theta_k^*}$$

  where the points $P = \{(\theta_1^*, \mu_1), (\theta_2^*, \mu_2), \ldots\}$ are spikes in a 2D Poisson process with rate measure:

$$c\mu^{-1}(1-\mu)^{c-1} d\mu \alpha H(d\theta)$$

- The beta process with $c = 1$ is the de Finetti measure for the IBP. When $c \neq 1$ we have a two parameter generalization of the IBP.
- This is an example of a *completely random measure*.
- A beta process *does not* have Beta distributed marginals.

[Hjort 1990, Ghahramani et al. 2007]

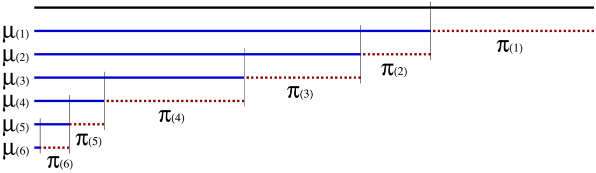
Stick-breaking Construction for Beta Processes

- When $c = 1$ it was shown that the following generates a draw of B:

$$v_k \sim \mathrm{Beta}(1, \alpha) \qquad \mu_k = (1 - v_k) \prod_{i=1}^{k-1} (1 - v_i) \qquad \theta_k^* \sim H$$

$$B = \sum_{k=1}^{\infty} \mu_k \delta_{\theta_k^*}$$

- The above is the complement of the stick-breaking construction for DPs!



[Teh et al. 2007]

Applications of Indian Buffet Processes

- The IBP can be used in concert with different likelihood models in a variety of applications.

$$Z \sim \mathrm{IBP}(\alpha) \qquad X \sim F(Z, Y)$$

$$Y \sim H \qquad p(Z, Y|X) = \frac{p(Z, Y)p(X|Z, Y)}{p(X)}$$

- Latent factor models for distributed representation [Griffiths and Ghahramani 2005].
- Matrix factorization for collaborative filtering [Meeds et al 2007].
- Latent causal discovery for medical diagnostics [Wood et al 2006].
- Protein complex discovery [Chu et al 2006].
- Psychological choice behaviour [Görür and Rasmussen 2006].
- Independent Components Analysis [Knowles and Ghahramani 2007].

Infinite Independent Components Analysis

- Each image X_i is a linear combination of sparse features:

$$X_i = \sum_k \Lambda_k^* y_{ik}$$

where y_{ik} is activity of feature k with sparse prior. One possibility is a mixture of a Gaussian and a point mass at 0:

$$y_{ik} = z_{ik} a_{ik} \qquad a_{ik} \sim \mathcal{N}(0, 1) \qquad Z \sim \text{IBP}(\alpha)$$

- An ICA model with infinite number of features.

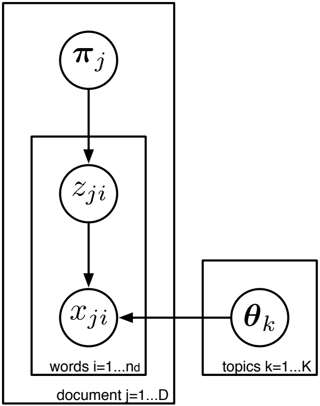
[Knowles and Ghahramani 2007]

Outline

Topic Modelling with Latent Dirichlet Allocation

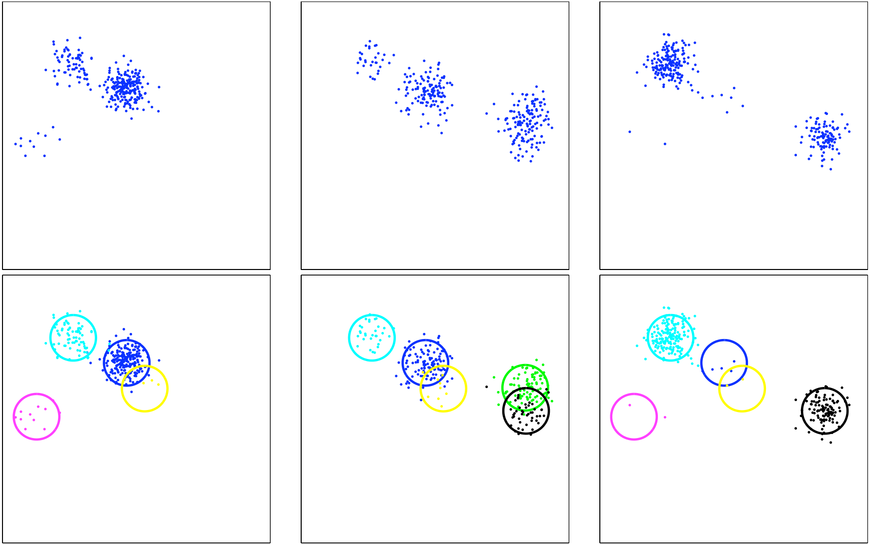
- Infer topics from a document corpus, topics being sets of words that tend to co-occur together.
- Using (Bayesian) latent Dirichlet allocation:

$$
\begin{aligned}
\boldsymbol{\pi}_j &\sim \text{Dirichlet}(\tfrac{\alpha}{K}, \ldots, \tfrac{\alpha}{K}) \\
\boldsymbol{\theta}_k &\sim \text{Dirichlet}(\tfrac{\beta}{W}, \ldots, \tfrac{\beta}{W}) \\
z_{ji} | \boldsymbol{\pi}_j &\sim \text{Multinomial}(\boldsymbol{\pi}_j) \\
x_{ji} | z_{ji}, \boldsymbol{\theta}_{z_{ji}} &\sim \text{Multinomial}(\boldsymbol{\theta}_{z_{ji}})
\end{aligned}
$$



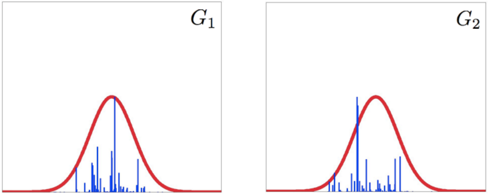
- Can we take $K \to \infty$?

Hierarchical Dirichlet Processes

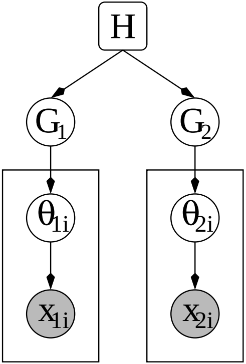


Hierarchical Dirichlet Processes

- Use a DP mixture for each group.



- Unfortunately there is no sharing of clusters across different groups because H is smooth.
- Solution: make the base distribution H discrete.
- Put a DP prior on the common base distribution.



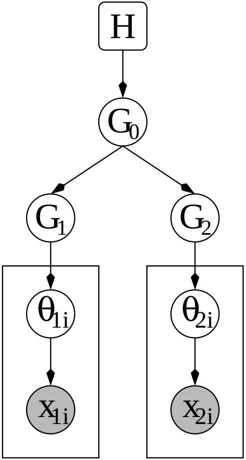
[Teh et al. 2006]

Hierarchical Dirichlet Processes

- A hierarchical Dirichlet process:

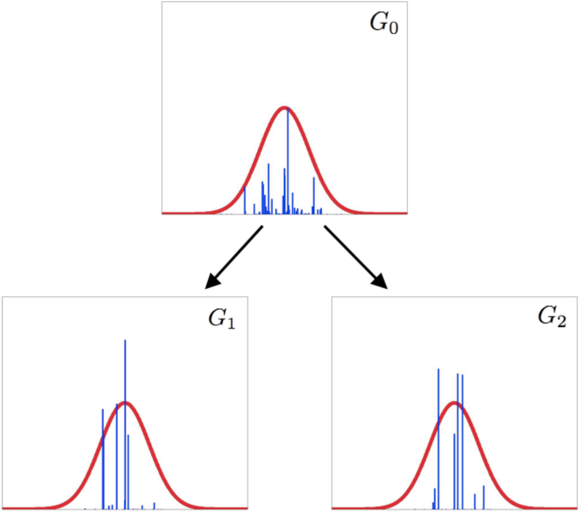
$$G_0 \sim \mathrm{DP}(\alpha_0, H)$$
$$G_1, G_2 | G_0 \sim \mathrm{DP}(\alpha, G_0) \text{ iid}$$

- Extension to larger hierarchies is straightforward.



Hierarchical Dirichlet Processes

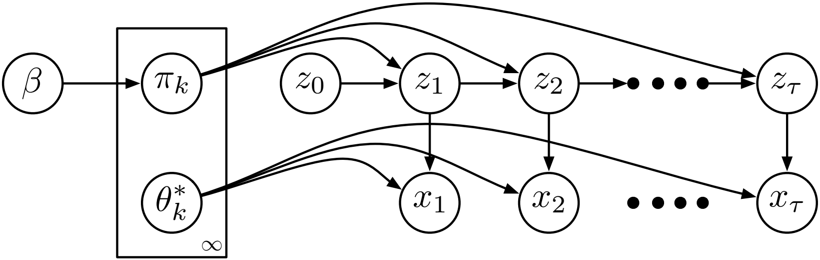
- Making G_0 discrete forces shared cluster between G_1 and G_2.



Hierarchical Dirichlet Processes

- Document topic modelling:
  - Allows documents to be modelled with DP mixtures of topics, with topics shared across corpora.
- Infinite hidden Markov modelling:
  - Allows HMMs with an infinite number of states, with transitions from each allowable state to every other allowable state.
- Learning discrete structures from data:
  - Determining number of objects, nonterminals, states etc.

Infinite Hidden Markov Models



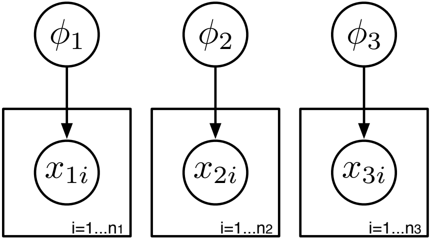
$\boldsymbol{\beta} \sim \text{GEM}(\gamma)$ $\quad$ $\boldsymbol{\pi}_k|\boldsymbol{\beta} \sim \text{DP}(\alpha, \boldsymbol{\beta})$ $\quad$ $z_i|z_{i-1}, \boldsymbol{\pi}_{z_{i-1}} \sim \text{Multinomial}(\boldsymbol{\pi}_{z_{i-1}})$

$\theta_k^* \sim H$ $\quad$ $x_i|z_i, \theta_{z_i}^* \sim F(\theta_{z_i}^*)$

- Hidden Markov models with an infinite number of states.
- Hierarchical DPs used to share information among transition probability vectors prevents "run-away" states.

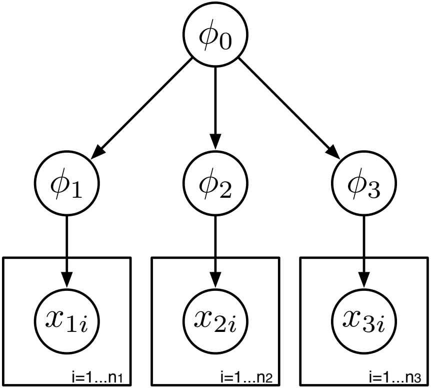
[Beal et al. 2002, Teh et al. 2006]

Hierarchical Modelling



- Better estimation of parameters.
- Multitask learning, learning to learn: generalizing across related tasks.

Hierarchical Modelling



- Better estimation of parameters.
- Multitask learning, learning to learn: generalizing across related tasks.

Outline

Pitman-Yor Processes

- Two-parameter generalization of the Chinese restaurant process:

$$p(\text{customer } n \text{ sat at table } k | \text{past}) = \begin{cases} \frac{n_k - \beta}{n-1+\alpha} & \text{if occupied table} \\ \frac{\alpha + \beta K}{n-1+\alpha} & \text{if new table} \end{cases}$$

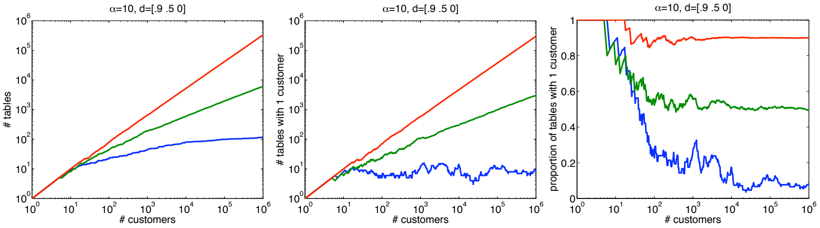
- Associating each cluster k with a unique draw $\theta_k^* \sim H$, the corresponding Pólya urn scheme is also exchangeable.
- De Finetti's Theorem states that there is a random measure underlying this two-parameter generalization.
  - This is the *Pitman-Yor process*.
- The Pitman-Yor process also has a stick-breaking construction:

$$\pi_k = v_k \prod_{i=1}^{k-1}(1 - v_i) \quad \beta_k \sim \text{Beta}(1 - \beta, \alpha + \beta k) \quad \theta_k^* \sim H \quad G = \sum_{k=1}^{\infty} \pi_k \delta_{\theta_k^*}$$

[Pitman and Yor 1997, Perman et al. 1992]

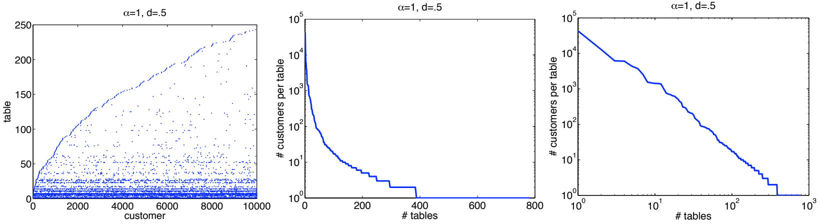
Pitman-Yor Processes

- Two salient features of the Pitman-Yor process:
  - With more occupied tables, the chance of even more tables becomes higher.
  - Tables with smaller occupancy numbers tend to have lower chance of getting new customers.
- The above means that Pitman-Yor processes produce Zipf's Law type behaviour, with $K = O(\alpha n^{\beta})$.

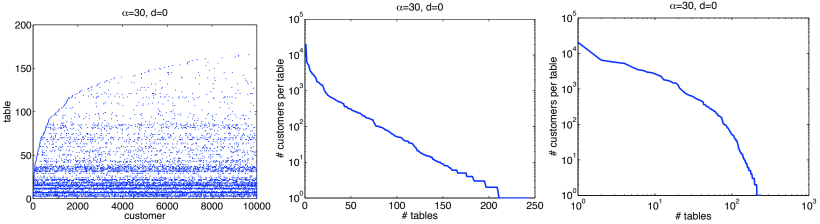


Pitman-Yor Processes

Draw from a Pitman-Yor process



Draw from a Dirichlet process



Hierarchical Pitman-Yor Language Models

- Pitman-Yor processes can be suitable models for many natural phenomena with power-law statistics.
- Language modelling with Markov assumption:

$$
\begin{aligned}
&p(\text{Mary has a little lamb}) \\
\approx &p(\text{Mary})p(\text{has}|\text{Mary})p(\text{a}|\text{Mary has})p(\text{little}|\text{has a})p(\text{lamb}|\text{a little})
\end{aligned}
$$

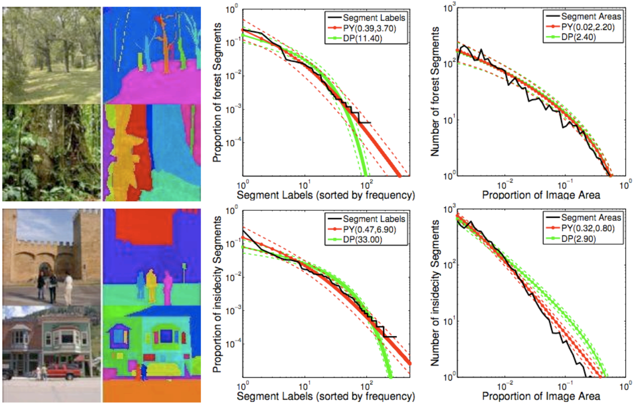
- Parameterize with $p(w_3|w_1, w_2) = G_{w_1,w_2}[w_3]$ and use a hierarchical Pitman-Yor process prior:

$$
\begin{aligned}
G_{w_1,w_2}|G_{w_2} &\sim \mathrm{PY}(\alpha_2, \beta_2, G_{w_2}) \\
G_{w_2}|G_{\emptyset} &\sim \mathrm{PY}(\alpha_1, \beta_1, G_{\emptyset}) \\
G_{\emptyset}|U &\sim \mathrm{PY}(\alpha_0, \beta_0, U)
\end{aligned}
$$

- State-of-the-art results, connection to Kneser-Ney smoothing.

[Goldwater et al. 2006a, Teh 2006b, Wood et al. 2009]

Image Segmentation with Pitman-Yor Processes



- Human segmentations of images also seem to follow power-law.
- An unsupervised image segmentation model based on dependent hierarchical Pitman-Yor processes achieves state-of-the-art results.

[Sudderth and Jordan 2009]

Outline

Summary

- Motivation for Bayesian nonparametrics:
  - Allows practitioners to define and work with models with large support, sidesteps model selection.
  - New models with useful properties.
  - Large variety of applications.
- Various standard Bayesian nonparametric models:
  - Dirichlet processes
  - Hierarchical Dirichlet processes
  - Infinite hidden Markov models
  - Indian buffet and beta processes
  - Pitman-Yor processes
- Touched upon two important theoretical tools:
  - Consistency and Kolmogorov's Consistency Theorem
  - Exchangeability and de Finetti's Theorem
- Described a number of applications of Bayesian nonparametrics.
- Missing: Inference methods based on MCMC, variational etc, consistency and convergence.

Other Introductions to Bayesian Nonparametrics

- Zoubin Gharamani, UAI 2005 Tutorial.
- Michael Jordan, NIPS 2005 Tutorial.
- Volker Tresp, ICML nonparametric Bayes workshop 2006.
- Peter Orbanz, Foundations of Nonparametric Bayesian Methods, 2009.
- I have given a number myself (check webpage).
- I have an introduction to Dirichlet processes [Teh 2007], and another to hierarchical Bayesian nonparametric models [Teh and Jordan 2010].

Bayesian Nonparametric Software

- Hierarchical Bayesian Compiler (HBC). Hal Daume III.
  http://www.cs.utah.edu/ hal/HBC/
- DPpackage. Alejandro Jara.
  http://cran.r-project.org/web/packages/DPpackage/index.html
- Hierarchical Pitman Yor Language Model. Songfang Huang.
  http://homepages.inf.ed.ac.uk/s0562315/progs/index.html
- Nonparametric Bayesian Mixture Models. Yee Whye Teh.
  http://www.gatsby.ucl.ac.uk/ ywteh/research/software.html
- Others...

Outline

Representations of Dirichlet Processes

- Posterior Dirichlet process:

$$\begin{aligned} G &\sim \mathrm{DP}(\alpha, H) \\ \theta | G &\sim G \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \theta &\sim H \\ G|\theta &\sim \mathrm{DP}\left(\alpha + 1, \tfrac{\alpha H + \delta_\theta}{\alpha + 1}\right) \end{aligned}$$

- Pólya urn scheme:

$$\theta_n | \theta_{1:n-1} \sim \frac{\alpha H + \sum_{i=1}^{n-1} \delta_{\theta_i}}{\alpha + n - 1}$$

- Chinese restaurant process:

$$p(\text{customer } n \text{ sat at table } k | \text{past}) = \begin{cases} \frac{n_k}{n-1+\alpha} & \text{if occupied table} \\ \frac{\alpha}{n-1+\alpha} & \text{if new table} \end{cases}$$

- Stick-breaking construction:

$$\pi_k = \beta_k \prod_{i=1}^{k-1} (1 - \beta_i) \qquad \beta_k \sim \mathrm{Beta}(1, \alpha) \qquad \theta_k^* \sim H \qquad G = \sum_{k=1}^{\infty} \pi_k \delta_{\theta_k^*}$$

Posterior Dirichlet Processes

- Suppose G is DP distributed, and θ is G distributed:

$$\begin{aligned} G &\sim \mathrm{DP}(\alpha, H) \\ \theta | G &\sim G \end{aligned}$$

- We are interested in:
  - The marginal distribution of θ with G integrated out.
  - The posterior distribution of G conditioning on θ.

Posterior Dirichlet Processes

Conjugacy between Dirichlet Distribution and Multinomial.

- Consider:

$$(\pi_1, \ldots, \pi_K) \sim \mathrm{Dirichlet}(\alpha_1, \ldots, \alpha_K)$$
$$z|(\pi_1, \ldots, \pi_K) \sim \mathrm{Discrete}(\pi_1, \ldots, \pi_K)$$

  z is a multinomial variate, taking on value $i \in \{1, \ldots, n\}$ with probability π_i.

- Then:

$$z \sim \mathrm{Discrete}\left(\frac{\alpha_1}{\sum_i \alpha_i}, \ldots, \frac{\alpha_K}{\sum_i \alpha_i}\right)$$
$$(\pi_1, \ldots, \pi_K)|z \sim \mathrm{Dirichlet}(\alpha_1 + \delta_1(z), \ldots, \alpha_K + \delta_K(z))$$

  where $\delta_i(z) = 1$ if z takes on value i, 0 otherwise.

- Converse also true.

Posterior Dirichlet Processes

- Fix a partition $(A_1, \ldots, A_K)$ of Θ. Then

$$(G(A_1), \ldots, G(A_K)) \sim \text{Dirichlet}(\alpha H(A_1), \ldots, \alpha H(A_K))$$
$$P(\theta \in A_i | G) = G(A_i)$$

- Using Dirichlet-multinomial conjugacy,

$$P(\theta \in A_i) = H(A_i)$$
$$(G(A_1), \ldots, G(A_K))|\theta \sim \text{Dirichlet}(\alpha H(A_1) + \delta_\theta(A_1), \ldots, \alpha H(A_K) + \delta_\theta(A_K))$$

- The above is true for every finite partition of Θ. In particular, taking a really fine partition,

$$p(d\theta) = H(d\theta)$$

  i.e. $\theta \sim H$ with G integrated out.

- Also, the posterior $G|\theta$ is also a Dirichlet process:

$$G|\theta \sim \text{DP}\left(\alpha + 1, \frac{\alpha H + \delta_\theta}{\alpha + 1}\right)$$

Posterior Dirichlet Processes

$$\begin{aligned} G &\sim \mathrm{DP}(\alpha, H) \\ \theta|G &\sim G \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \theta &\sim H \\ G|\theta &\sim \mathrm{DP}\left(\alpha + 1, \tfrac{\alpha H + \delta_\theta}{\alpha + 1}\right) \end{aligned}$$

Pólya Urn Scheme

- First sample:

$$\begin{aligned} & \theta_1 | G \sim G && G \sim \mathrm{DP}(\alpha, H) \\ \iff \quad & \theta_1 \sim H && G|\theta_1 \sim \mathrm{DP}(\alpha+1, \tfrac{\alpha H + \delta_{\theta_1}}{\alpha+1}) \end{aligned}$$

- Second sample:

$$\begin{aligned} & \theta_2 | \theta_1, G \sim G && G|\theta_1 \sim \mathrm{DP}(\alpha+1, \tfrac{\alpha H + \delta_{\theta_1}}{\alpha+1}) \\ \iff \quad & \theta_2|\theta_1 \sim \tfrac{\alpha H + \delta_{\theta_1}}{\alpha+1} && G|\theta_1, \theta_2 \sim \mathrm{DP}(\alpha+2, \tfrac{\alpha H + \delta_{\theta_1} + \delta_{\theta_2}}{\alpha+2}) \end{aligned}$$

- n^{th} sample

$$\begin{aligned} & \theta_n | \theta_{1:n-1}, G \sim G && G|\theta_{1:n-1} \sim \mathrm{DP}(\alpha+n-1, \tfrac{\alpha H + \sum_{i=1}^{n-1} \delta_{\theta_i}}{\alpha+n-1}) \\ \iff \quad & \theta_n|\theta_{1:n-1} \sim \tfrac{\alpha H + \sum_{i=1}^{n-1} \delta_{\theta_i}}{\alpha+n-1} && G|\theta_{1:n} \sim \mathrm{DP}(\alpha+n, \tfrac{\alpha H + \sum_{i=1}^{n} \delta_{\theta_i}}{\alpha+n}) \end{aligned}$$

Stick-breaking Construction

- Returning to the posterior process:

$$\begin{aligned} G &\sim \mathrm{DP}(\alpha, H) \\ \theta|G &\sim G \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \theta &\sim H \\ G|\theta &\sim \mathrm{DP}(\alpha + 1, \tfrac{\alpha H + \delta_\theta}{\alpha + 1}) \end{aligned}$$

- Consider a partition $(\theta, \Theta \backslash \theta)$ of Θ. We have:

$$\begin{aligned} (G(\theta), G(\Theta \backslash \theta))|\theta &\sim \mathrm{Dirichlet}((\alpha + 1)\tfrac{\alpha H + \delta_\theta}{\alpha + 1}(\theta), (\alpha + 1)\tfrac{\alpha H + \delta_\theta}{\alpha + 1}(\Theta \backslash \theta)) \\ &= \mathrm{Dirichlet}(1, \alpha) \end{aligned}$$

- G has a point mass located at θ:

$$G = \beta \delta_\theta + (1 - \beta)G' \qquad \text{with} \qquad \beta \sim \mathrm{Beta}(1, \alpha)$$

  and G' is the (renormalized) probability measure with the point mass removed.
- What is G'?

Stick-breaking Construction

- Currently, we have:

$$\begin{aligned} G &\sim \mathrm{DP}(\alpha, H) \\ \theta &\sim G \end{aligned} \quad \Rightarrow \quad \begin{aligned} \theta &\sim H \\ G|\theta &\sim \mathrm{DP}(\alpha+1, \tfrac{\alpha H + \delta_\theta}{\alpha+1}) \\ G &= \beta\delta_\theta + (1-\beta)G' \\ \beta &\sim \mathrm{Beta}(1, \alpha) \end{aligned}$$

- Consider a further partition $(\theta, A_1, \ldots, A_K)$ of Θ:

$$\begin{aligned} &(G(\theta), G(A_1), \ldots, G(A_K)) \\ =&(\beta, (1-\beta)G'(A_1), \ldots, (1-\beta)G'(A_K)) \\ \sim&\, \mathrm{Dirichlet}(1, \alpha H(A_1), \ldots, \alpha H(A_K)) \end{aligned}$$

- The agglomerative/decimative property of Dirichlet implies:

$$(G'(A_1), \ldots, G'(A_K))|\theta \sim \mathrm{Dirichlet}(\alpha H(A_1), \ldots, \alpha H(A_K))$$
$$G' \sim \mathrm{DP}(\alpha, H)$$

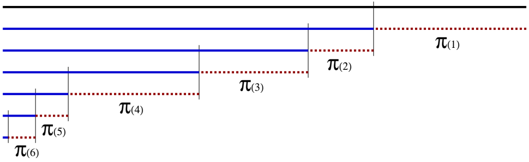
Stick-breaking Construction

- We have:

$$
\begin{aligned}
G &\sim \mathrm{DP}(\alpha, H) \\
G &= \beta_1 \delta_{\theta_1^*} + (1-\beta_1) G_1 \\
G &= \beta_1 \delta_{\theta_1^*} + (1-\beta_1)(\beta_2 \delta_{\theta_2^*} + (1-\beta_2) G_2) \\
&\vdots \\
G &= \sum_{k=1}^{\infty} \pi_k \delta_{\theta_k^*}
\end{aligned}
$$

where

$$\pi_k = \beta_k \prod_{i=1}^{k-1}(1-\beta_i) \qquad \beta_k \sim \mathrm{Beta}(1, \alpha) \qquad \theta_k^* \sim H$$



Outline

Stick-breaking Construction

- We shall assume the following HDP hierarchy:

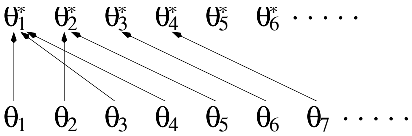
$$\begin{aligned} G_0 &\sim \mathrm{DP}(\gamma, H) \\ G_j | G_0 &\sim \mathrm{DP}(\alpha, G_0) \quad \text{for } j = 1, \dots, J \end{aligned}$$

- The stick-breaking construction for the HDP is:

$$\begin{aligned} G_0 &= \textstyle\sum_{k=1}^{\infty} \pi_{0k} \delta_{\theta_k^*} & \theta_k^* &\sim H \\ \pi_{0k} &= \beta_{0k} \textstyle\prod_{l=1}^{k-1} (1 - \beta_{0l}) & \beta_{0k} &\sim \mathrm{Beta}\left(1, \gamma\right) \\ G_j &= \textstyle\sum_{k=1}^{\infty} \pi_{jk} \delta_{\theta_k^*} & & \\ \pi_{jk} &= \beta_{jk} \textstyle\prod_{l=1}^{k-1} (1 - \beta_{jl}) & \beta_{jk} &\sim \mathrm{Beta}\left(\alpha\beta_{0k}, \alpha(1 - \textstyle\sum_{l=1}^{k} \beta_{0l})\right) \end{aligned}$$

Hierarchical Pòlya Urn Scheme

- Let $G \sim \mathrm{DP}(\alpha, H)$.
- We can visualize the Pòlya urn scheme as follows:



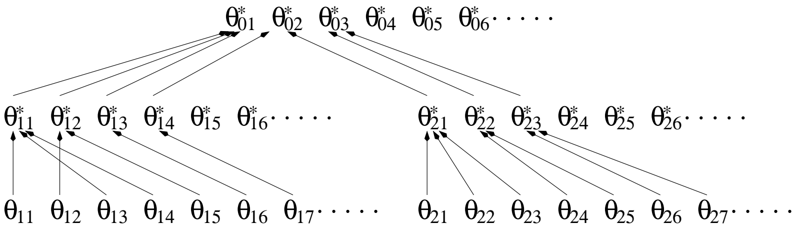
where the arrows denote to which θ^*_k each θ_i was assigned and

$$\theta_1, \theta_2, \ldots \sim G \text{ i.i.d.}$$
$$\theta^*_1, \theta^*_2, \ldots \sim H \text{ i.i.d.}$$

(but $\theta_1, \theta_2, \ldots$ are not independent of $\theta^*_1, \theta^*_2, \ldots$).

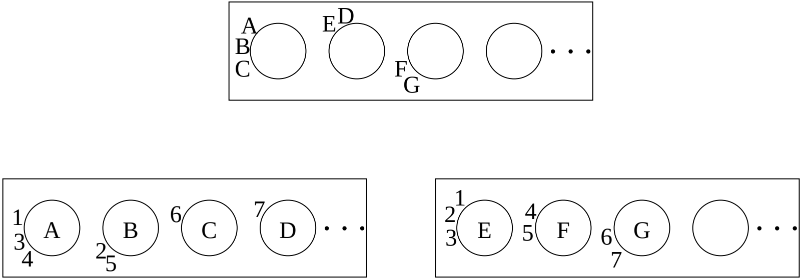
Hierarchical Pòlya Urn Scheme

- Let $G_0 \sim \mathrm{DP}(\gamma, H)$ and $G_1, G_2 | G_0 \sim \mathrm{DP}(\alpha, G_0)$.
- The hierarchical Pòlya urn scheme to generate draws from G_1, G_2:



Chinese Restaurant Franchise

- Let $G_0 \sim \mathrm{DP}(\gamma, H)$ and $G_1, G_2 | G_0 \sim \mathrm{DP}(\alpha, G_0)$.
- The Chinese restaurant franchise describes the clustering of data items in the hierarchy:



Outline

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Bibliography I

Dirichlet Processes and Beyond in Machine Learning

Dirichlet Processes were first introduced by [Ferguson 1973], while [Antoniak 1974] further developed DPs as well as introduce the mixture of DPs. [Blackwell and MacQueen 1973] showed that the Pólya urn scheme is exchangeable with the DP being its de Finetti measure. Further information on the Chinese restaurant process can be obtained at [Aldous 1985, Pitman 2002]. The DP is also related to Ewens' Sampling Formula [Ewens 1972]. [Sethuraman 1994] gave a constructive definition of the DP via a stick-breaking construction. DPs were rediscovered in the machine learning community by [Neal 1992, Rasmussen 2000].

Hierarchical Dirichlet Processes (HDPs) were first developed by [Teh et al. 2006], although an aspect of the model was first discussed in the context of infinite hidden Markov models [Beal et al. 2002]. HDPs and generalizations have been applied across a wide variety of fields.
Dependent Dirichlet Processes are sets of coupled distributions over probability measures, each of which is marginally DP [MacEachern et al. 2001]. A variety of dependent DPs have been proposed in the literature since then [Srebro and Roweis 2005, Griffin 2007, Caron et al. 2007]. The infinite mixture of Gaussian processes of [Rasmussen and Ghahramani 2002] can also be interpreted as a dependent DP.
Indian Buffet Processes (IBPs) were first proposed in [Griffiths and Ghahramani 2006], and extended to a two-parameter family in [Ghahramani et al. 2007]. [Thibaux and Jordan 2007] showed that the de Finetti measure for the IBP is the beta process of [Hjort 1990], while [Teh et al. 2007] gave a stick-breaking construction and developed efficient slice sampling inference algorithms for the IBP.
Nonparametric Tree Models are models that use distributions over trees that are consistent and exchangeable. [Blei et al. 2004] used a nested CRP to define distributions over trees with a finite number of levels. [Neal 2001, Neal 2003] defined Dirichlet diffusion trees, which are binary trees produced by a fragmentation process. [Teh et al. 2008] used Kingman's coalescent [Kingman 1982b, Kingman 1982a] to produce random binary trees using a coalescent process. [Roy et al. 2007] proposed annotated hierarchies, using tree-consistent partitions first defined in [Heller and Ghahramani 2005] to model both relational and featural data.

Markov chain Monte Carlo Inference algorithms are the dominant approaches to inference in DP mixtures. [Neal 2000] is a good review of algorithms based on Gibbs sampling in the CRP representation. Algorithm 8 in [Neal 2000] is still one of the best algorithms based on simple local moves. [Ishwaran and James 2001] proposed blocked Gibbs sampling in the stick-breaking representation instead due to the simplicity in implementation. This has been further explored in [Porteous et al. 2006]. Since then there has been proposals for better MCMC samplers based on proposing larger moves in a Metropolis-Hastings framework [Jain and Neal 2004, Liang et al. 2007a], as well as sequential Monte Carlo [Fearnhead 2004, Mansingkha et al. 2007].
Other Approximate Inference Methods have also been proposed for DP mixture models. [Blei and Jordan 2006] is the first variational Bayesian approximation, and is based on a truncated stick-breaking representation. [Kurihara et al. 2007] proposed an

improved VB approximation based on a better truncation technique, and using KD-trees for extremely efficient inference in large scale applications. [Kurihara et al. 2007] studied improved VB approximations based on integrating out the stick-breaking weights. [Minka and Ghahramani 2003] derived an expectation propagation based algorithm. [Heller and Ghahramani 2005] derived tree-based approximation which can be seen as a Bayesian hierarchical clustering algorithm. [Daume III 2007] developed admissible search heuristics to find MAP clusterings in a DP mixture model.

Computer Vision and Image Processing. HDPs have been used in object tracking [Fox et al. 2006, Fox et al. 2007b, Fox et al. 2007a]. An extension called the transformed Dirichlet process has been used in scene analysis [Sudderth et al. 2006b, Sudderth et al. 2006a, Sudderth et al. 2008], a related extension has been used in fMRI image analysis [Kim and Smyth 2007, Kim 2007]. An extension of the infinite hidden Markov model called the nonparametric hidden Markov tree has been introduced and applied to image denoising [Kivinen et al. 2007a, Kivinen et al. 2007b].
Natural Language Processing. HDPs are essential ingredients in defining nonparametric context free grammars [Liang et al. 2007b, Finkel et al. 2007]. [Johnson et al. 2007] defined adaptor grammars, which is a framework generalizing both probabilistic context free grammars as well as a variety of nonparametric models including DPs and HDPs. DPs and HDPs have been used in information retrieval [Cowans 2004], word segmentation [Goldwater et al. 2006b], word morphology modelling [Goldwater et al. 2006a], coreference resolution [Haghighi and Klein 2007], topic modelling [Blei et al. 2004, Teh et al. 2006, Li et al. 2007]. An extension of the HDP called the hierarchical Pitman-Yor process has been applied to language modelling [Teh 2006a, Teh 2006b, Goldwater et al. 2006a].[Savova et al. 2007] used annotated hierarchies to construct syntactic hierarchies. Theses on nonparametric methods in NLP include [Cowans 2006, Goldwater 2006].
Other Applications. Applications of DPs, HDPs and infinite HMMs in bioinformatics include [Xing et al. 2004, Xing et al. 2007, Xing et al. 2006, Xing and Sohn 2007a, Xing and Sohn 2007b]. DPs have been applied in relational learning [Shafto et al. 2006, Kemp et al. 2006, Xu et al. 2006], spike sorting [Wood et al. 2006a, Görür 2007]. The HDP has been used in a cognitive model of categorization [Griffiths et al. 2007]. IBPs have been applied to infer hidden causes [Wood et al. 2006b], in a choice model [Görür et al. 2006], to modelling dyadic data [Meeds et al. 2007], to overlapping clustering [Heller and Ghahramani 2007], and to matrix factorization [Wood and Griffiths 2006].

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