Nonparametric stick breaking priors with simple weights

Ramsés H. Mena

IIMAS-UNAM

(work with Fuentes-García, R., Ruggiero, M. and Walker, S.G.)

Gatsby Computational Neuroscience Unit

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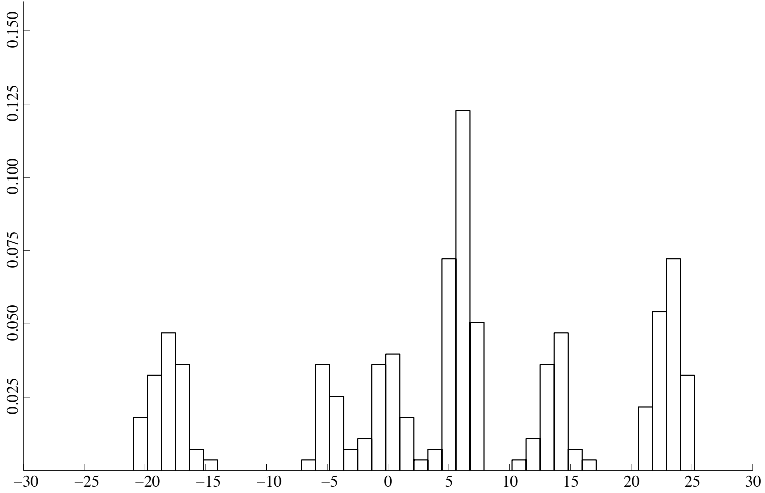
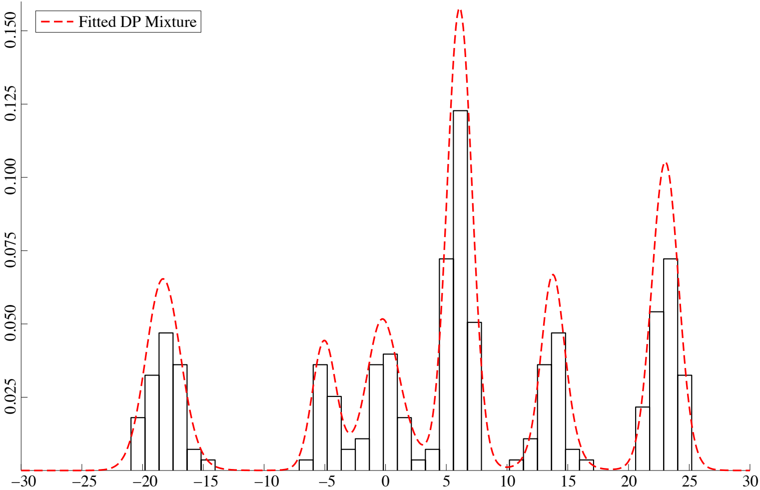
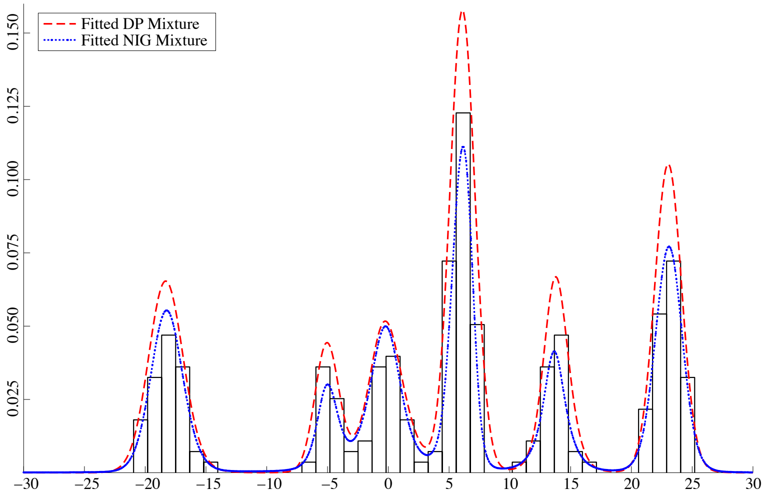
- Suppose we observe the following data



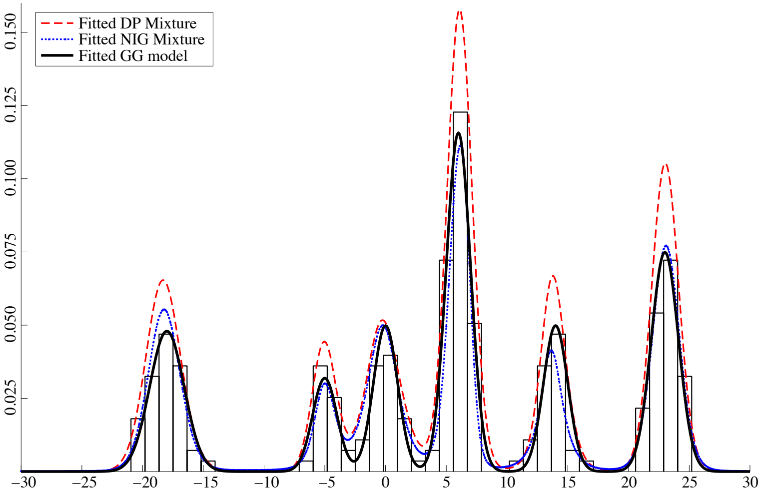
- we could fit of DP mixture $\mathsf{f}(\cdot) = \int_{\mathbb{X}} f(\cdot \mid x)\mu(\mathrm{d}x)$, $\mu \sim \mathcal{D}_{\theta\nu_0}$



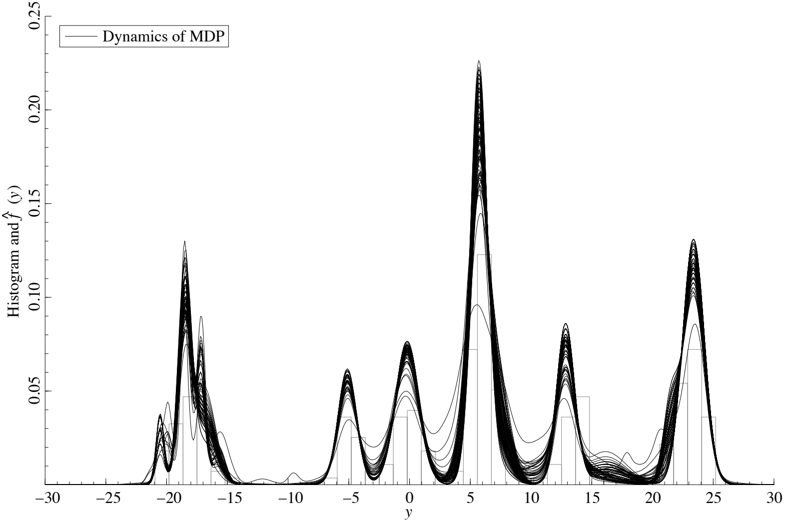
- ...or alternatively a NIG mixture model



- . . . or even a more elaborated GG mixture model



- These estimators are result of a convergent MCMC



→A convergent state of these MCMC estimators typically needs:

- Hyper-parameters specifications in the kernel $f(\cdot \mid x)$ and ν_0
- Randomization of the parameters of RPMs μ
- Techniques to accelerate and attain convergence

→ "General" RPMs partially ease some of these aspects, however there is a tractability issue:

The more general the rpm the less manageable it becomes

Here we present a simplistic approach that addresses some of these issues and explore its applications in depending settings

1. Motivation
2. Geometric weights
3. Dependent processes
4. Estimation

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Stick breaking weights

- Any discrete dist. can be represented as

$$P(B) = \sum_{i=1}^{\infty} \mathsf{w}_i \, \delta_{z_i}(B), \quad B \in \mathcal{X}, \qquad \sum_i \mathsf{w}_i = 1$$

- Make the "weights", $(\mathsf{w}_i)_{i\geq 1}$, and "locations", $(z_i)_{i\geq 1}$ random

  $\Rightarrow \mu$ is a Random Prob. Measure (RPM)

- Stick-breaking weights

$$\mathsf{w}_1 = \mathsf{V}_1, \qquad \mathsf{w}_i = \mathsf{V}_i \prod_{j<i} (1 - \mathsf{V}_j), \quad i \geq 2$$

- Let $(\mathsf{V}_i)_{i\geq 1}$ indep. $[0,1]$-valued r.v.'s with $\mathsf{E}[\sum_{i\geq 1} \log(1 - \mathsf{V}_i)] = -\infty$

Stick breaking weights

- Any discrete dist. on a Polish space $(\mathbb{X}, \mathcal{X})$ can be represented as

$$\mu(B) = \sum_{i=1}^{\infty} \mathsf{w}_i \, \delta_{z_i}(B), \quad B \in \mathcal{X}, \qquad \sum_i \mathsf{w}_i = 1 \text{ a.s.}$$

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Dirichlet process $\mathcal{D}_{\theta,\nu_0}$

- Sethuraman (1994)

$$\text{if } \mathsf{V}_i \overset{\text{iid}}{\sim} \mathsf{Be}(1,\theta) \qquad \text{and} \qquad z_i \overset{\text{iid}}{\sim} \nu_0 \quad (\text{indep. of } \mathsf{V}_i\text{'s})$$

- μ follows Ferguson (1973) Dirichlet process ($\mu \sim \mathcal{D}_{\theta,\nu_0}$)

  i.e. a stochastic processes, $\{\mu(B)\}_{B\in\mathcal{X}}$, with finite dim. dist.

$$(\mu(B_1),\ldots,\mu(B_k)) \sim \mathsf{Dirichlet}(\theta\nu_0(B_1),\ldots,\theta\nu_0(B_k))$$

  for all $k \geq 1$ and all partitions $(B_1,\ldots,B_k)$ of $\mathbb{X}$.

Some basic properties of $\mathcal{D}_\alpha$

- $\mathsf{E}[\mu(B)] = \nu_0(B)$, $\mathsf{Var}[\mu(B)] = \frac{\nu_0(B)(1-\nu_0(B))}{\theta+1}$

  $\mathsf{Cov}(\mu(B_2), \mu(B_2)) = \frac{\nu_0(B_1 \cap B_2) - \nu_0(B_1)\nu_0(B_2)}{\theta+1}$

If $X_i \mid \mu \overset{\text{iid}}{\sim} \mu$ and $\mu \sim \mathcal{D}_{\theta,\nu_0}$, hence $X_i \sim \nu_0$, forall $i = 1, 2, \ldots$

$$\mu \mid X_1, \ldots, X_n \sim \mathcal{D}_{\theta\nu_0 + n\mu_n} \quad \text{(Conjugate posterior)}$$

with $\mu_n = n^{-1} \sum_{i=1}^n \delta_{X_i}$

$$\mathsf{E}[\mu \mid X_1, \ldots, X_n] = \frac{\theta}{\theta+n}\nu_0 + \frac{n}{\theta+n}\sum_{i=1}^n \frac{\delta_{X_i}}{n}, \quad \text{(Bayes estimator)}$$

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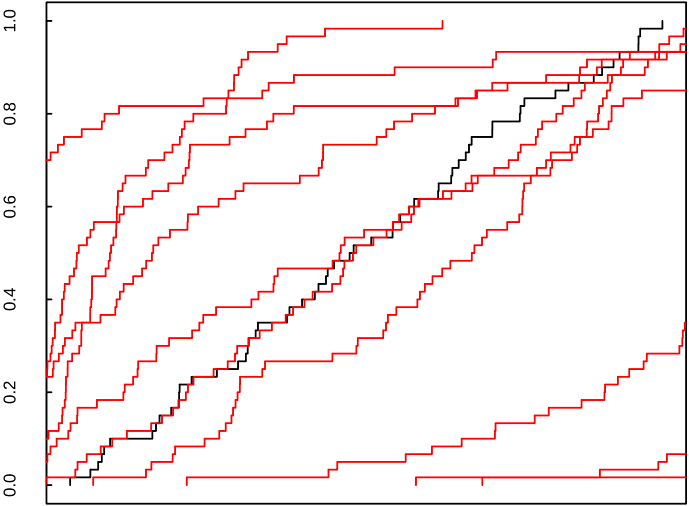
Precision parameter θ



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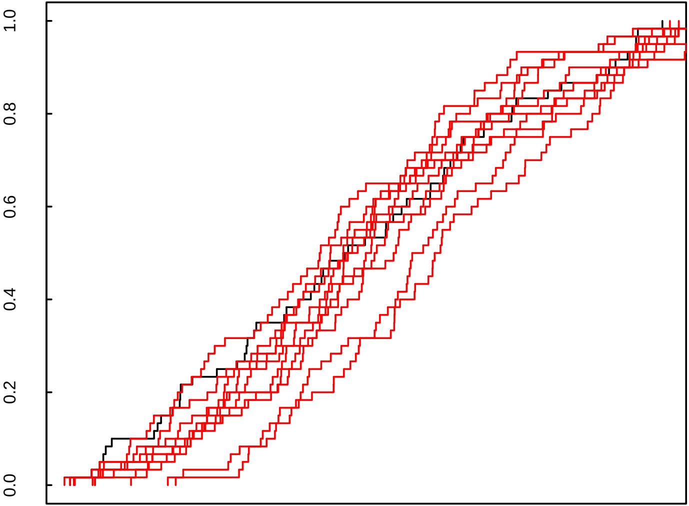

θ can be seen as a precision param.

Clustering induced by $\mathcal{D}_\alpha$

- Since $\mathcal{D}_\alpha$ a.s. discrete, $P(X_i = X_j) > 0$ for $i \neq j$
- $(X_1, \ldots, X_n)$ can be encoded to $(X_1^*, \ldots, X_{K_n}^*)$ unique values
- with random frequencies $(N_1, \ldots, N_{K_n})$, i.e. $\sum_{i=1}^{K_n} N_i = n$
- The support of $(N_1, \ldots, N_{K_n})$ is in bijection with

$$\mathscr{P}_{[n]} := \text{Set of all partitions of } \{1, \ldots, n\}$$

- Selecting $\mathcal{D}_\alpha$ induces an Exchangeable Partition Probability Function (EPPF) –Ewens (1972) and Antoniak (1974)–

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$$\mathbb{P}(\text{Obs. in } k \text{ groups with freq. } n_1, \ldots, n_k) = \frac{\theta^k}{(\theta)_n} \prod_{j=1}^{k} (n_j - 1)!$$

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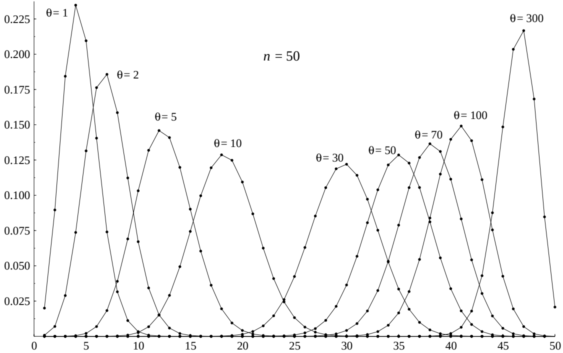
Clustering induced by $\mathcal{D}_\alpha$

- Summing over all posible partitions for fixed k

$$\mathbb{P}(K_n = k) = \frac{\theta^k}{(\theta)_n} |s(n,k)|$$

where $s(n,k)$ for $n \geq k \geq 1$ Stirling numbers of the first type.

The precision param. θ also controls the grouping.
Too informative!



Distribution of K_n, when $\mu \sim \mathcal{D}_{(\theta\nu_0)}$

BNP mixtures

For continuous data use μ-mixtures

BNP mixture models

$$Y_i \mid \mathsf{f} \overset{\text{iid}}{\sim} \mathsf{f} \quad \text{where} \quad \mathsf{f}(\cdot) = \int_{\mathbb{X}} f(\cdot \mid x)\mu(\mathrm{d}x)$$

$\mathsf{f}(\cdot)$ random density (Lo 84': $\mathsf{Q} = \mathcal{D}_\alpha$)

Density estimation & Clustering problems

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$$
\begin{aligned}
Y_i \mid X_i &\overset{\text{ind}}{\sim} f(Y_i \mid X_i) \qquad i \geq 1 \ (e.g.\ \mathsf{f}(\cdot)\ \text{Leb. density}) \\
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Equivalently

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BNP mixtures: Density estimation

A Bayes density estimator, *e.g.*

$$\mathsf{E}\left[\mathsf{f}(y) \mid Y^{(n)}\right] = \sum_{k=1}^{n} \int_{\mathbb{X}} f(y \mid x) \sum_{\mathbf{p}_k \in \mathscr{P}_{[n]}^k} \mathsf{E}\left[\mu(\mathrm{d}x) \mid x_{1:k}^*\right] \mathbb{P}[x_{1:k}^* \in \mathbf{p}_k \mid Y^{(n)}]$$

where $x_{1:k}^* = (x_1^*, \ldots, x_k^*)$ and $\mathbf{p}_k \in \mathscr{P}_{[n]}^k$

- $\mathsf{E}\left[\mu(\mathrm{d}x) \mid x_{1:k}^*\right]$ denotes the predictive
  - For large n virtually impossible to evaluate exactly
  - The need of MCMC methods is evident

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BNP mixtures: Posterior distribution on $\mathscr{P}_{[n]}$

- Posterior clustering under BNP mixture (or clustering likelihood!)

$$\mathbb{P}[\mathbf{p}_k \mid Y^{(n)}] \propto \Pi_k^{(n)}(n_1, \ldots, n_k) \prod_{j=1}^{k} \int_{\mathbb{X}} \prod_{i \in \mathcal{J}_j} f(y_i \mid x_i) \nu_0(\mathrm{d}x_i)$$

where as before $\mathbf{p}_k \in \mathscr{P}_{[n]}^k$ and $\mathcal{J}_j := \{i : X_i = X_j^*\}, \quad j = 1, \ldots, k$

▷ No longer exchangeable due to effect of $f(\cdot \mid x)$ the y's

- Summing over all the partitions for fixed k we obtain the posterior on the number of groups of size $k = 1, \ldots, n$

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BNP mixtures: Toy example (10 data points)

- $f(y \mid \theta) = \mathsf{N}(y \mid \mu, \lambda^{-1})$, $\mu \sim \mathcal{D}_{\theta\nu_0}$

  $\nu_0(\mathrm{d}\mu, \mathrm{d}\lambda) = \mathsf{N}(\mu \mid 0, \frac{10}{\lambda})\mathsf{Exp}(\lambda \mid 1)\mathrm{d}\mu\mathrm{d}\lambda$

▷ $\mathbf{p}_2 = \{\{y_1, \dots, y_4\}, \{y_5, \dots, y_{10}\}\}$

→ integer partition $(n_1, n_2) = (4, 6)$

▷ If $\theta = 1$ posterior mode is at $\mathbf{p}_2$ with $\mathbb{P}[\mathbf{p}_2 \mid y^{(n)}] = 0.332$

▷ Posterior on #groups: mode at $k = 3$

with $\mathbb{P}[K_{10} = 3 \mid y^{(n)}] = 0.39$ & $\mathbb{P}[K_{10} = 2 \mid y^{(n)}] = 0.37$

▷ If $\theta = 0.5$: $\mathbb{P}[K_{10} = 3 \mid y^{(n)}] = 0.31$ & $\mathbb{P}[K_{10} = 2 \mid y^{(n)}] = 0.59$ – $\mathsf{E}(K_{10}) = 2.1$ –

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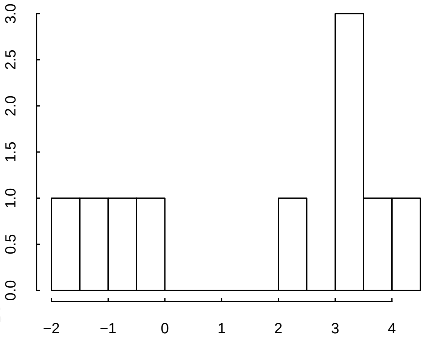

BNP mixtures: Toy example (10 data points)

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▷ $\mathbf{p}_2 = \{\{y_1, \ldots, y_4\}, \{y_5, \ldots, y_{10}\}\}$

→ integer partition $(n_1, n_2) = (4, 6)$

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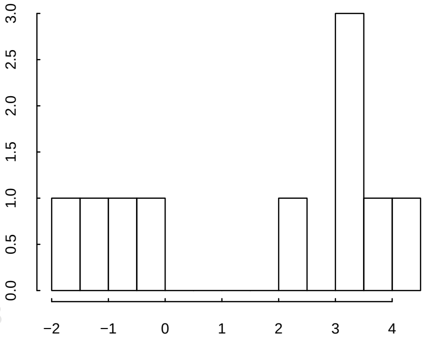

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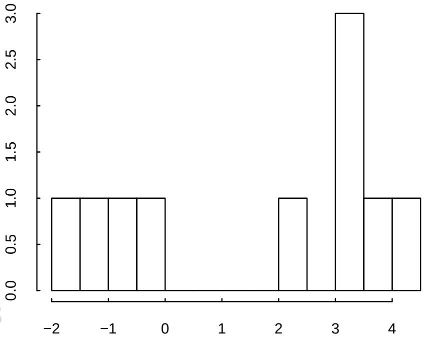

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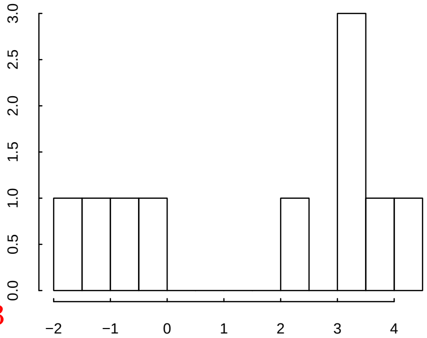

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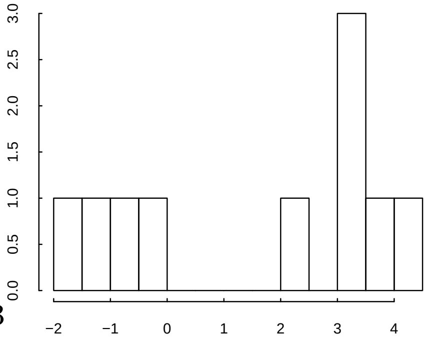

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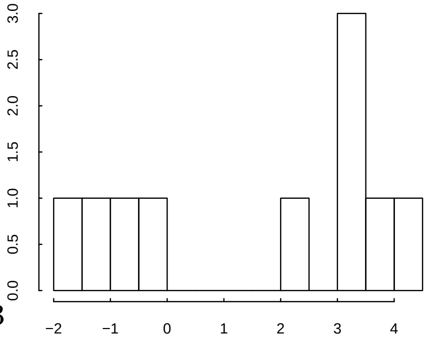

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A simplified RPM: Geometric weights

- Given that for the $\mathcal{D}_{\theta\nu_0}$ a randomization of θ is needed we could instead consider the simplified RPM

$$\mu(B) = \sum_{i=1}^{\infty} \mathsf{E}[\mathsf{w}_i]\delta_{z_i}(B) = \sum_{i=1}^{\infty} \lambda(1-\lambda)^{i-1}\delta_{z_i}(B)$$

where $\lambda = (\theta+1)^{-1}$ and $\lambda \sim \mathsf{Be}(a,b)$, *i.e.* with geometric weights.

▷ Namely, a DP with the randomness of the weights removed!

▷ This RPM has ordered weights!

▷ Still has full support wrt weak topology

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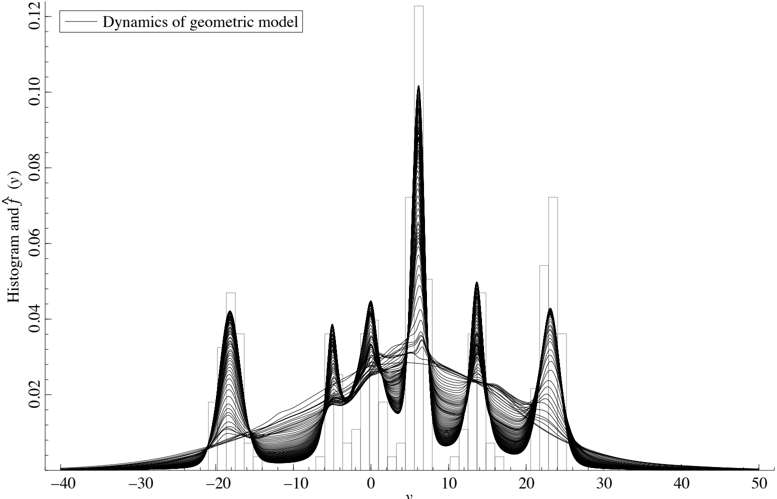
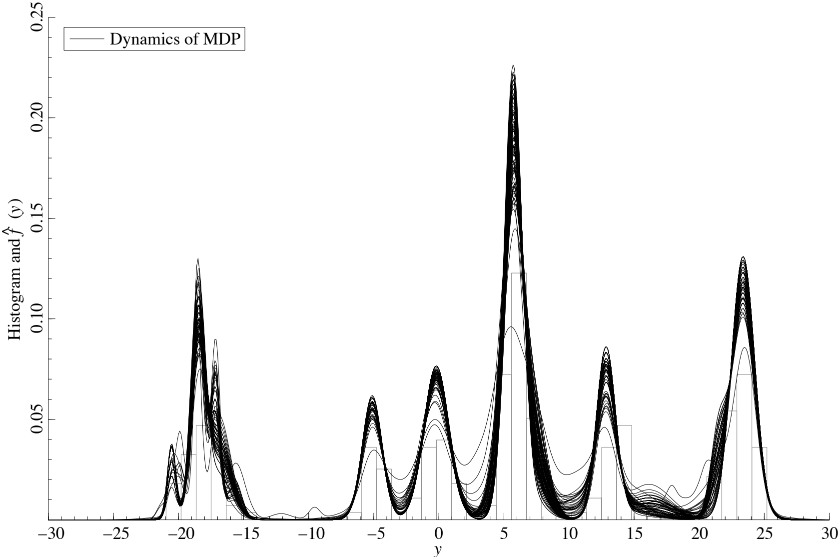
100 iter. BNP mixture model based on geom. weights

$$f(y) = \int_{\mathbb{X}} f(y \mid z)\mu(\mathrm{d}z) = \sum_{l \geq 1} \lambda(1-\lambda)^{l-1} f(y \mid \theta_l)$$



DP mixture


Dynamics of MDP
Histogram and $\hat{f}(y)$
0.25
0.20
0.15
0.10
0.05
–30
–25
–20
–15
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0
5
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25
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MCMC methods: via slice sampler (Walker 07')

$$\mathsf{f}(y) = \sum_{i=1}^{\infty} \mathsf{w}_i f(y \mid z_i) \qquad (*)$$

▷ Infinite summation becomes a problem since w_i's are not ordered

- Augment (*) through a uniform latent variable

$$f(y,u) = \sum_{j=1}^{\infty} \mathbb{I}(u < \mathsf{w}_j) f(y \mid z_i)$$

- Given u the set $A_u := \{j : \mathsf{w}_j > u\}$ is finite.

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Random set A_u

- So A_u is a finite subset of the set of positive integers
- For the DP weights the A_u typically generates set of integers with gaps, e.g. $\{2, 5, 16, 40, 200, 3029\}$
- But given that the representation

$$\mu(B) = \sum_{i=1}^{\infty} \mathsf{w}_i \, \delta_{z_i}(B), \quad B \in \mathcal{X}$$

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A different construction of the weights

Consider the random density defined by

$$f(y \mid A) = \frac{1}{\#A} \sum_{j \in A} f(y \mid z_i)$$

with A a finite random subset of $\mathbb{N}_+$

- Here we look at $A = \{1, \ldots, N\}$ with $N \sim q_N$ so

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$$\mathsf{w}_i = \sum_{N=i}^{\infty} \frac{q_N}{N}$$

q_N a prob. mass function on $\mathbb{N}_+$

- Weights are ordered!

For example if q_N is a $\mathsf{Neg-Bin}(r, \lambda)$ we get

$$\mathsf{w}_i = \frac{1}{i}\binom{i+r-2}{r-1}\lambda^r(1-\lambda)^{i-1}{}_2\mathsf{F}_1(1, i+r-1; i+1; \lambda)$$

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Dependent processes

What happens with a different type of dependence?

Namely, we have observations typically capture with models such as:

- $X_{n+1} = \phi X_n + \varepsilon_t$
- $dX_t = a(X_t, \theta)dt + \sigma(X_t, \theta)dW_t$
- $X_i = f(\mathbf{Z}, \beta)$
- etc..

We still want to be nonparametric!

- Nonparametric dependent random measures, *i.e*

$$\{\mu_n\}_{n=0}^{\infty}, \quad \{\mu_t\}_{t\geq 0}, \quad \{\mu_z\}_{z\in Z}$$

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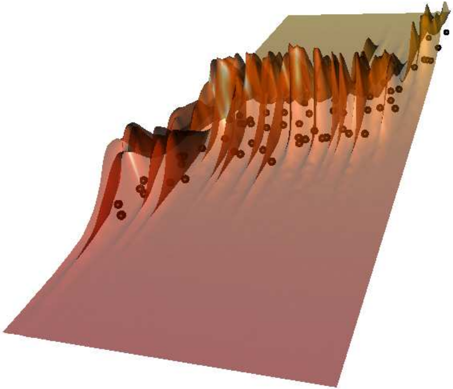
Covariate dependent

- Introduce dependence through $\{\lambda_z\}_{z\in\mathcal{Z}}$

$$\lambda_z = \frac{e^{\xi(z)}}{1+e^{\xi(z)}}, \qquad \{\xi(z)\} \sim \mathrm{GP}(\mu,\sigma)$$



$$\eta_z := \int y\, f_z(y) \mathrm{d}y$$

$$f_z(y) = \sum_{l\geq 1} \lambda_z(1-\lambda_z)^{l-1} f(y \mid \theta_l)$$

Let's look at a continuous time dependent NP process.

$$\mu(t) = \sum_{i \geq 0} w_i(t)\, \delta_{x_i(t)}$$

where, for each $i \geq 0$, $\{w_i(t)\}_{t\geq 0}$, $\{x_i(t)\}_{t\geq 0}$ are certain *ad hoc* stochastic processes.

- In general we might think $\mu(t)$ inherits some of the continuity and stability properties of the processes $\{w_i(t)\}$ and $\{x_i(t)\}$

Geometric stick-breaking process

Definition

Let $\{\mu(t), t \geq 0\}$ a stochastic process with values on $\mathcal{P}_{\mathbb{X}}$ defined on $(\Omega, \mathscr{F}, \mathbb{P})$ such that for each $t \geq 0$

$$\mu(t) = \lambda_t \sum_{i \geq 0} (1 - \lambda_t)^{i-1} \, \delta_{x_i}$$

where ν_0 is an non-atomic distribution on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ and $\{\lambda_t\}_{t \geq 0}$ is a diffusion process with paths in $\mathcal{C}_{[0,1]}([0, \infty))$ and infinitesimal generator

$$\mathcal{A} = \left[\frac{c}{a+b-1}(a - (a+b)\lambda)\right] \frac{\mathrm{d}}{\mathrm{d}\lambda} + \frac{c}{a+b-1}\lambda(1-\lambda)\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}$$

with domain $\mathscr{D}(\mathcal{A}) = \mathcal{C}^2([0,1])$. We name $\{\mu(t), t \geq 0\}$ the Geometric Stick Breaking process with parameters (a, b, c, ν_0) denoted by $\mathrm{GSB}(a, b, c, \nu_0)$

Geometric stick-breaking process

- $\{\lambda_t\}_{t\geq 0}$ is a diffusion process with the following features:
  - Stationary with invariant distribution $\mathsf{Be}(a, b)$
  - Reversible
  - When $c := (a + b - 1)/2 \Rightarrow \{\lambda_t\}_{t\geq 0}$ Wright-Fisher model

Which of these properties are inherited by $\mu_t \sim \text{GSBP}(a, b, c, \nu_0)$?

- Let $\mathscr{P}_g(\mathbb{X}) \subset \mathcal{P}_{\mathbb{X}}$ the set of purely atomic probability measures on $\mathbb{X}$

Propiedades GSB(a, b, c, ν_0)

Proposition

Let $\{\mu_t\}_{t\geq 0}$ a GSB(a, b, c, ν_0) process. Then, $\{\mu_t\}_{t\geq 0}$ has an infinitesimal generator given by

$$\mathcal{B}\varphi_m(\mu) = \left(\frac{a}{2}(1-\lambda) - \frac{b}{2}\lambda\right) \sum_{i_1,\dots,i_m\geq 1} f(x_{i_1},\dots,x_{i_m}) \frac{\partial}{\partial\lambda} h(\lambda; m, i_1, \dots, i_m)$$
$$+ \frac{1}{2}\lambda(1-\lambda) \sum_{i_1,\dots,i_m\geq 1} f(x_{i_1},\dots,x_{i_m}) \frac{\partial^2}{\partial\lambda^2} h(\lambda; m, i_1, \dots, i_m)$$

with domain

$$\mathscr{D}(\mathcal{B}) = \left\{\varphi \in C(\mathscr{P}_g(\mathbb{X})) :\ \varphi = \varphi_m(\mu) = \langle f, \mu^m\rangle,\ f \in C(\mathbb{X}^m),\ m \in \mathbb{N}\right\}$$

and where

$$h(\lambda; m, i_1, \dots, i_m) = \lambda_t^m (1-\lambda_t)^{\sum_{j=1}^m i_j - m}.$$

Properties of $\mathrm{GSB}(a, b, c, \nu_0)$

Proposition

Let $\mathbb{X}$ be a Polish space, $\{\mu_t\}_{t\geq 0}$ a $\mathrm{GSB}(a, b, c, \nu_0)$ process on $\mathscr{P}_g(\mathbb{X})$. Hence $\{\mu_t\}_{t\geq 0}$ is a Feller process with trajectories on $\mathcal{C}_{\mathscr{P}_g(\mathbb{X})}([0, \infty))$.

Proposition

Let $\mathbb{X}$ be a Polish space, $\{\mu_t\}_{t\geq 0}$ a $\mathrm{GSB}(a, b, c, \nu_0)$ process on $\mathscr{P}_g(\mathbb{X})$. Hence $\{\mu_t\}_{t\geq 0}$ is reversible and strictly stationary.

Summing up, $\{\mu_t\}_{t\geq 0}$ is a diffusion process with values in the space of purely atomic probability measures, with continuous trajectories, stationary and reversible!

Mixtures of GSB(a, b, c, ν_0) process

- If we require that the process takes values on $\mathscr{P}_c(\mathbb{X}) \subset \mathcal{P}_{\mathbb{X}}$ (all continuous prob. measures), we consider

$$f_t(y) = \int_{\mathbb{X}} f(y \mid z)\mu_t(\mathrm{d}z) = \sum_{l \geq 1} \lambda_t(1-\lambda_t)^{l-1} f(y \mid \theta_l)$$

where $f(\cdot \mid \theta)$ is a well defined Lebesgue density and $\theta_l \overset{\text{iid}}{\sim} \nu_0$, ν_0 non-atomic.

Estimation for single trajectory data

Sup. we observe only one trayectory $\{y_{t_i}\}_{i=1}^n$ and we use the mixture model. In hierarchical notation

$$\begin{aligned} y_i \mid t_i, x_i &\sim f(\cdot \mid x_i) \\ \{x_i\} &\sim \mu_t \\ \mu_t &\sim \mathrm{GSB}(a, b, c, \nu_0). \end{aligned} \tag{1}$$

where $x_i := x_{t_i}$.

- We will estimate this model through a Gibbs sampler algorithm

Diffusion part $\{\lambda_t\}$

The transition density for $\{\lambda_t\}$ can be expressed as

$$p(\lambda_t \mid \lambda_0) = \sum_{m=0}^{\infty} \mathsf{q}_t(m)\, D(\lambda_t | m, \lambda_0)$$

where

$$\mathsf{q}_t(m) = \frac{(a+b)_m\, e^{-m\,c\,t}}{m!} (1 - e^{-c\,t})^{a+b}$$

and

$$D(\lambda_t | m, \lambda_0) = \sum_{k=0}^{m} \mathsf{Be}(\lambda_t | a+k, b+m-k)\, \mathsf{Bin}(k | m, \lambda_0).$$

(M. and Walker, 2009)

Diffusion part $\{\lambda_t\}$

- Sup. we have observations (t_i, s_i), where

$$s_i \mid \lambda_i \sim \text{Geom}(\lambda_i)$$
$$(\lambda_1, \ldots, \lambda_n) \sim \text{WF}(a, b, c)$$

With the *fidis* for $\{\lambda_t\}$ given by

$$p(\lambda_1, \ldots, \lambda_n) = p(\lambda_0) \prod_{i=1}^{n} p(\lambda_i \mid \lambda_{i-1}), \qquad \text{where } \lambda_i := \lambda_{t_i}$$

and $p(\lambda_0) = \mathsf{Be}(\lambda_0 \mid a, b)$

$p(\lambda_i \mid \lambda_{i-1})$ has an infinite summation $\Rightarrow$ slice it!

$$p(\lambda_t \mid \lambda_0) = \sum_{m=0}^{\infty} \frac{g(m)}{g(m)} \mathsf{q}_t(m)\, D(\lambda_t | m, \lambda_0)$$

where g is a decreasing func. with known inverse, *e.g.* $g(m) = e^{-m}$

Diffusion part $\{\lambda_t\}$

- Augment the transition density via the latent variables $(u_i, d_i, k_i)_{i=1}^n$

$$p(\lambda_i, u_i, k_i, d_i \mid \lambda_{i-1}) =$$

$$\mathbf{1}(u_i < g(d_i))\,\frac{\mathsf{q}_i(d_i)}{g(d_i)}\,\mathsf{Be}(\lambda_i|a+k_i, b+d_i-k_i)\,\mathsf{Bin}(k_i|d_i,\lambda_{i-1})$$

Hence, the likelihood for the "complete data" is

$$l(a,b,c) = \mathrm{Beta}(\lambda_0|a,b)\prod_{i=1}^n p(\lambda_i,u_i,k_i,d_i|\lambda_{i-1})\,\lambda_i(1-\lambda_i)^{s_i-1}$$

If we assume priors for $a, b, c \overset{\text{iid}}{\sim} \mathrm{Exp}(1)$ then the posterior distributions $\pi(a \mid b, c, \ldots) \propto l(a,b,c)e^{-a}$, etc. are log-concave, *e.g.*

$$\log\pi(c|a,b,\ldots) = \sum_{i=1}^n \left\{(a+b)\log(1-e^{-c\,\tau_i}) - d_i\,c\,\tau_i\right\} - c + C,$$

Condicionales completas

$$\pi(k_i|\ldots) \propto \binom{d_i}{k_i} \frac{\mathbf{1}(k_i \in \{0,1,\ldots,d_i\})}{\Gamma(a+k_i)\Gamma(b+d_i-k_i)} \left\{ \frac{\lambda_i \lambda_{i-1}}{(1-\lambda_i)(1-\lambda_{i-1})} \right\}^{k_i}$$

easy to sample as it takes a finite number of values

$$\pi(u_i \mid \ldots) = \mathrm{U}_{[0,g(d_i)]}(u_i)$$

$$\pi(d_i|\ldots) \propto \frac{\Gamma(a+d+d_i)^2 e^{d_i[1-c\tau_i]}\, \mathbf{1}(k_i \le d_i \le -\log u_i)}{\Gamma(b+d_i-k_i)\Gamma(d_i-k_i+1)\{(1-\lambda_{i-1})(1-\lambda_i)\}^{-d_i}}$$

Also finite due to the u_i's

Complete conditionals

The complete conditionals for λ_i, $i \neq 0, n$, are given by

$$\pi(\lambda_i | \ldots) = \mathrm{Beta}(1 + a + k_i + k_{i+1}, s_i - 1 + b + d_i + d_{i+1} - k_i - k_{i+1}),$$

and

$$\pi(\lambda_0 | \ldots) = \mathrm{Beta}(a + k_1, b + d_1 - k_1)$$

and

$$\pi(\lambda_n | \ldots) = \mathrm{Beta}(1 + a + k_n, s_n - 1 + b + d_n - k_n).$$

This procedure via the latent variables could also be useful to estimate other diffusion processes

Gibbs sampler

For the remaining part of the model we use a similar idea "slice"

- That is, we "augment" the model

$$y_i|t_i,\lambda_i,\theta \sim \sum_{l=1}^{\infty} \lambda_i(1-\lambda_i)^{l-1} f(y_i|\theta_l),$$

with two random variables (s_i, v_i) and $\{\psi_l\}$ (a seq. of decreasing numbers s.t. $\{l : \psi_l > v\}$ is a known set), i.e.

$$y_i, v_i, s_i|\lambda_i,\theta \sim \psi_{s_i}^{-1}\mathbf{1}(v_i < \psi_{s_i})\,\lambda_i(1-\lambda_i)^{s_i-1}\,f(y_i|\theta_{s_i}).$$

In this way

$$\begin{aligned}
\pi(s_i|\ldots) &\propto \psi_{s_i}^{-1}\lambda_i(1-\lambda_i)^{s_i-1}\,f(y_i|\theta_{s_i})\,\mathbf{1}(s_i \in \{l : \psi_l > v_i\}) \\
\pi(v_i|\ldots) &= \mathrm{U}_{(0,\psi_{s_i})}(v_i) \\
\pi(\theta_l|\ldots) &\propto \prod_{s_i=l} f(y_i|\theta_l)\,g_0(\theta_l) \qquad \text{for } l = 1,\ldots,\max_i\{l : \psi_l > v_i\}
\end{aligned}$$

Gibbs sampler

Summarizing, we need

- $\pi(a \mid b, c, \ldots)$, $\pi(b \mid a, c, \ldots)$ and $\pi(c \mid a, b, \ldots)$ (via ARS)
- $\pi(k_i| \ldots)$, $\pi(u_i| \ldots)$ y $\pi(d_i| \ldots)$ (via Inverse CDF)
- $\pi(\lambda_i| \ldots)$ (Beta's)
- $\pi(s_i| \ldots)$ and $\pi(v_i| \ldots)$ (via Inverse CDF)
- $\pi(\theta_i| \ldots)$ (if f y g_0 are conjugated ✓, otherwise via ARMS, M-H, etc)

A bit long, but only a very simple Gibbs sampler

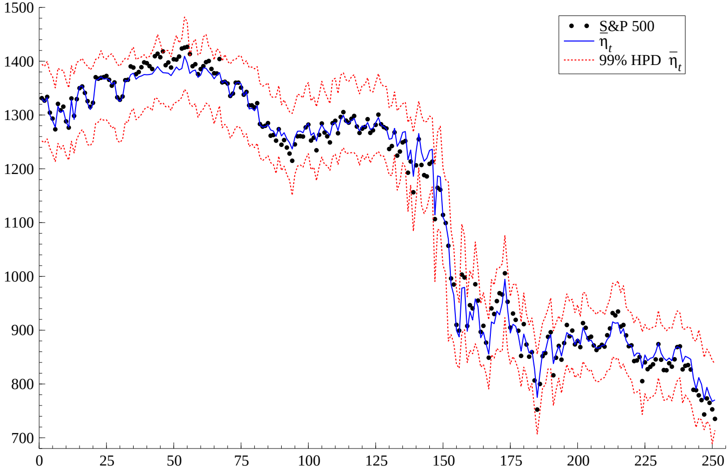


Figura: MC estimator for $\bar{\eta}_t$ (solid) and corresponding 99 % highest posterior density intervals (dotted) for the S&P 500 data set (dots). The estimates are based on 10000 iterations of the Gibbs sampler algorithm after 2000 iterations of burn in.

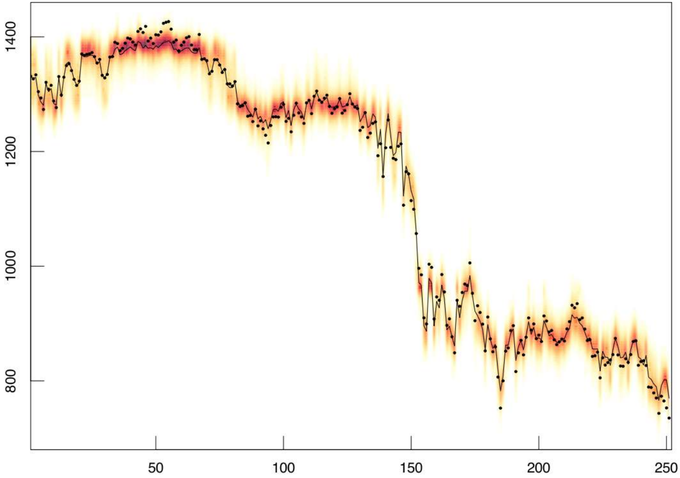


Figura: MCMC density estimator for the random density process, $\hat{f}_t$, (heat contour), mean of mean functional $\bar{\eta}_t$ (solid) for the S&P 500 data set (dots). The estimates are based on 10000 effective

EPPF

$$\Pi_k^n(n_1, n_2, \ldots, n_k) = \left(\frac{\lambda}{1-\lambda}\right)^n \sum_{(*)_k} (1-\lambda)^{\sum_{l=1}^k n_l j_l}$$

Then, one can obtains results such as

$$\mathsf{E}[K_n] = \sum_{r=1}^{n} (-1)^{r-1} \binom{n}{r} \frac{\lambda^r}{1-(1-\lambda)^r}$$

when k is large

$$\Pi_k^n(n_1, n_2, \ldots, n_k) \approx \left(\frac{\lambda}{1-\lambda}\right)^n (1-\lambda)^{n_{(1)}+2n_{(2)}+\cdots+kn_{(k)}}$$

(M. and Walker, 2012)

Thanks !

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