Matrix Tree Theorem (Kirchhoff 1847)

$\Omega = \{ \text{functional subgraphs} \}$

$A_\sigma = \{ \text{functional subgraphs containing cycle } \sigma \}$

$$\#\text{ spanning trees} = |\Omega| - \sum_{\sigma} |A_\sigma| + \sum_{\sigma_1, \sigma_2} |A_{\sigma_1} \cap A_{\sigma_2}| - \sum_{\sigma_1 \sigma_2 \sigma_3} |A_{\sigma_1} \cap A_{\sigma_2} \cap A_{\sigma_3}| \ldots$$

diagonal

$$= \prod_{i \neq r} \deg^+(i) + \sum_{\sigma} \operatorname{sign}(\sigma) \prod_i Q_{i\sigma(i)} + \sum_{\sigma_1, \sigma_2} \operatorname{sign}(\sigma_1, \sigma_2) \prod_i Q_{i, \sigma_1 \sigma_2(i)} + \ldots$$

$$= \det(Q)$$

Inclusion - Exclusion

$$A_1, A_2, \ldots A_n \subseteq \Omega$$

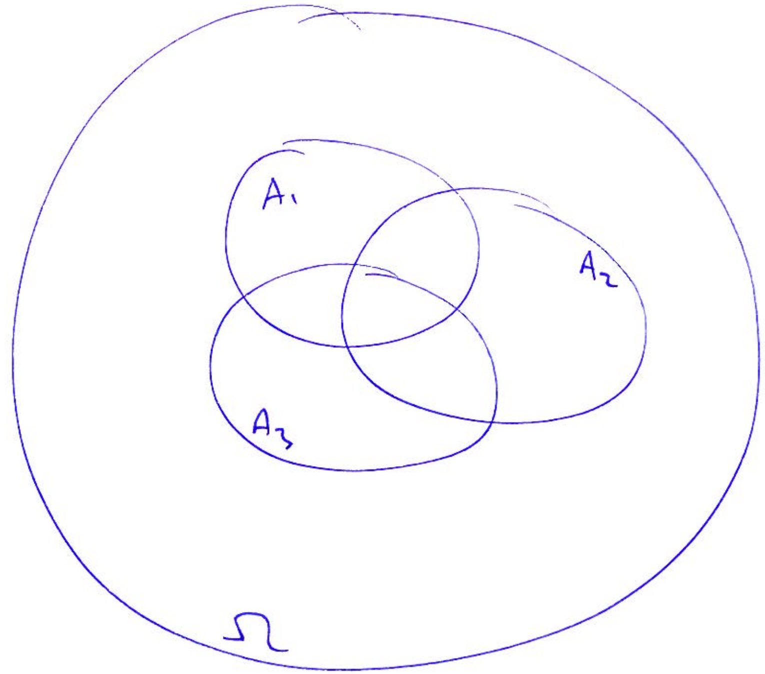
$$\#(A_1 \cup \ldots \cup A_n)^c = \#\{x \in \Omega : x \notin A_i \; \forall i\}$$

$$= S_0 - S_1 + S_2 - S_3 \ldots$$

$$S_0 = |\Omega|$$

$$S_1 = \sum_i |A_i|$$

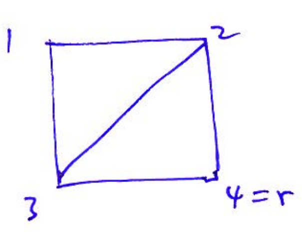
$$S_2 = \sum_{i,j} |A_i \cap A_j|$$

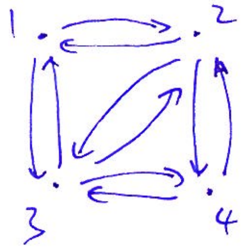


$$\#(A_1 \cup A_2 \cup A_3)^c = |\Omega| - \sum_i |A_i| + \sum_{ij} |A_i \cap A_j| - \sum_{ijk} |A_i \cap A_j \cap A_k|$$

Digraph

Directed edges, but no self-directed edges.



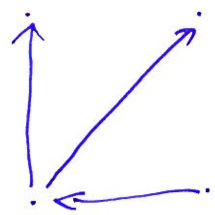


$$L = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 0 & 1 & 1 & -2 \end{pmatrix} \qquad Q_{44} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & -3 \end{pmatrix} \qquad |Q_{44}| = 8$$

Rooted Tree at $r=4$

A tree where all edges lead out from r.

$\boxed{\text{\# spanning trees of graph} = \text{\# \sout{root} spanning trees rooted at } r.}$



Functional ^sub^Graph

Each node picks ~~a~~ one parent, except r.

$\boxed{\text{spanning Rooted trees at } r \text{ are functional subgraphs}}$

$\boxed{\text{Functional subgraphs can contain cycles}}$

$\boxed{\text{\# spanning trees} = \text{\# functional subgraphs (rooted at } r) \text{ without cycles}}$

Permutations and Symmetric Group

$S_n = \{\text{permutations on } n \text{ objects}\}$ forms a group.

Each $\pi \in S_n$ is a bijective ~~functional~~ function $\pi : \{1 \ldots n\} \to \{1 \ldots n\}$, operator is composition.

$\pi = (1\ 2\ 3)(4\ 5)(6)$

$\boxed{\text{Every } \pi \text{ can be decomposed into disjoint cycles } \pi = \sigma_1 \circ \ldots \circ \sigma_k}$

$\boxed{\text{Every cycle can be decomposed into transpositions.}}$

$(1\ 2\ 3) = (1\ 2)(2\ 3)$

$2 \to 3$
$3 \to 2 \to 1$
$1 \to 2$

~~# transpositions of π~~ $=$ sign(π), π even if sign$(\pi) = 1$ # transpositions is even sign$(\pi) = -1$ if odd

$$\boxed{\mathrm{Det}(Q) = \sum_{\pi} \mathrm{sign}(\pi) \prod_{i=1}^{n} Q_{i,\pi(i)}}$$

Functional ~~S~~ubgraphs

\# functional subgraphs containing disjoint cycles $\sigma_1 \dots \sigma_k$ is $\prod_{i=1}^{n} |Q_{i\pi(i)}|$, $\pi = \sigma_1 \dots \sigma_k$.

sign is $+1$ if k even, $-$ if odd

[illegible]

if i not in cycle, # choices $= \deg^+(i) = Q_{i\pi(i)} = Q_{ii}$

if i in cycle, $\to j$, # choices $=$ # edges $j \to i = Q_{ij} = Q_{i\pi(i)}$

$$\boxed{\#\ \text{functional subgraphs} = \prod_{i=1}^{n} |Q_{i\pi(i)}|.}$$

$$\mathrm{sign}(\pi) \prod_{i=1}^{n} Q_{i\pi(i)} = \underbrace{\Big(\prod_{\substack{i \text{ not} \\ \text{in cycle}}} Q_{ii}\Big)}_{\text{positive}} \prod_{\text{cycle } \sigma_k} \mathrm{sign}(\sigma_k) \prod_{i \in \sigma_k} Q_{i,\sigma_k(i)}$$

$|\sigma_k| = \text{even}$

σ_k odd: negative ~~positive~~ ~~positive~~ positive

even ~~odd~~: positive negative.

$\underbrace{\qquad\qquad}_{\text{negative}}$

$$\mathrm{sign} = \begin{cases} - & \#\ \text{cycles odd} \\ + & \#\ \text{cycles even} \end{cases}$$

④