1. Let X_1, X_2, X_3, X_4, X_5 be independent with $X_i \sim \mathcal{N}(0,1)$. Define recursively

$$Y_1 \leftarrow X_1,\; Y_2 \leftarrow X_2{+}Y_1,\; Y_3 \leftarrow 2X_3{+}Y_2,\; Y_4 \leftarrow X_4{+}Y_3,\; Y_5 \leftarrow X_5{+}2Y_4.$$

(a) Find the covariance matrix Σ of Y;
First rewrite all the variables in terms of X_i:

$$Y_1 = X_1, \quad Y_2 = X_1 + X_2, \quad Y_3 = X_1 + X_2 + 2X_3$$

$$Y_4 = X_1 + X_2 + 2X_3 + X_4, \quad Y_5 = 2X_1 + 2X_2 + 4X_3 + 2X_4 + X_5.$$

From this we readily decuce the covariance matrix

$$\Sigma_Y = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 2 & 4 \\ 1 & 2 & 6 & 6 & 12 \\ 1 & 2 & 6 & 7 & 14 \\ 2 & 4 & 12 & 14 & 29 \end{pmatrix}.$$

(b) Find the concentration matrix $K = \Sigma^{-1}$ of Y.
Using the recursive definition we find the joint density to be

$$f(y) \propto e^{-\{y_1^2+(y_2-y_1)^2+(y_3-y_2)^2/4+(y_4-y_3)^2+(y_5-2y_4)^2\}/2}.$$

Expanding the squares and identifying coefficients we find the concentration matrix to be

$$K_Y = \frac{1}{4}\begin{pmatrix} 8 & -4 & 0 & 0 & 0 \\ -4 & 5 & -1 & 0 & 0 \\ 0 & -1 & 5 & -4 & 0 \\ 0 & 0 & -4 & 20 & -8 \\ 0 & 0 & 0 & -8 & 4 \end{pmatrix}.$$

(c) Find a directed acyclic graph $\mathcal{D}$ so that Y is directed Markov w.r.t. $\mathcal{D}$. This is a Markov chain: $1 \to 2 \to 3 \to 4 \to 5$.

(d) Construct the dependence graph of Y;
From the concentration matrix we deduce directly that the dependence graph is a line: $1 \sim 2 \sim 3 \sim 4 \sim 5$.
This result can also be obtained by interpreting the recursive definition as a directed Markov specification, and then realising that the corresponding DAG was perfect, so arrows could be changed to lines.

(e) Find the conditional distribution of Y_3 given $Y_1 = y_1, Y_2 = y_2, Y_4 = y_4, Y_5 = y_5$.
From the formulae for conditional distributions given in terms of the concentration matrices we get

$$\mathbf{V}(Y_3 \mid y_1, y_2, y_4, y_5) = 1/k_{33} = 4/5$$

and

$$\mathbb{E}(Y_3 \mid y_1, y_2, y_4, y_5) = -(k_{13}y_1 - k_{32}y_2 - k_{34}y_4 - k_{35}y_5)/k_{33} = (y_2 + 4y_4)/5.$$

This can be obtained in many other ways, but probably not any simpler, when the concentration matrix is available already.

2. Consider a Gaussian distribution $\mathcal{N}_4(0, \Sigma)$ with $K = \Sigma^{-1}$ satisfying the conditional independence restrictions of the graph $\mathcal{G} = (V, E)$ with $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}\}$. Assume that the following Wishart matrix has been observed, with 10 degrees of freedom:

$$\begin{pmatrix} 5 & 1 & 4 & 4 \\ 1 & 10 & 2 & 5 \\ 4 & 2 & 10 & 2 \\ 4 & 5 & 2 & 8 \end{pmatrix}.$$

(a) Perform one full cycle of the IPS algorithm to find the MLE of the concentration matrix, starting with $K = I$.

First we calculate that

$$10 \begin{pmatrix} 5 & 1 \\ 1 & 10 \end{pmatrix}^{-1} = \begin{pmatrix} 100/49 & -10/49 \\ -10/49 & 50/49 \end{pmatrix}$$

so the first update yields

$$T_{12}K = \begin{pmatrix} 100/49 & -10/49 & 0 & 0 \\ -10/49 & 50/49 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$10 \begin{pmatrix} 10 & 2 \\ 2 & 10 \end{pmatrix}^{-1} = \begin{pmatrix} 25/24 & -5/24 \\ -5/24 & 25/24 \end{pmatrix}$$

so the second update becomes

$$T_{12}T_{23}K = \begin{pmatrix} 100/49 & -10/49 & 0 & 0 \\ -10/49 & 25/24 + 1/49 & -5/24 & 0 \\ 0 & -5/24 & 25/24 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

etc., etc.

(b) Assume next that $K = \Sigma^{-1}$ satisfies the conditional independence restrictions of the graph $\mathcal{G}^* = (V, E^*)$ with $V = \{1, 2, 3, 4\}$ and $E^* = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$. Find the maximum likelihood estimate of the concentration matrix.

First we calculate that

$$10 \begin{pmatrix} 5 & 1 \\ 1 & 10 \end{pmatrix}^{-1} = \begin{pmatrix} 100/49 & -10/49 \\ -10/49 & 50/49 \end{pmatrix}$$

$$10 \begin{pmatrix} 10 & 2 \\ 2 & 10 \end{pmatrix}^{-1} = \begin{pmatrix} 25/24 & -5/24 \\ -5/24 & 25/24 \end{pmatrix}$$

$$10 \begin{pmatrix} 10 & 2 \\ 2 & 8 \end{pmatrix}^{-1} = \begin{pmatrix} 20/19 & -5/19 \\ -5/19 & 25/19 \end{pmatrix}$$

so that we get

$$\hat{K} = \begin{pmatrix} 100/49 & -10/49 & 0 & 0 \\ -10/49 & 50/49 + 25/24 - 1 & -20/96 & 0 \\ 0 & -20/96 & 25/24 + 20/19 - 1 & -5/19 \\ 0 & 0 & -5/19 & 25/19 \end{pmatrix}$$

which, not coincidentally, is the same result that one gets after the third step in the IPS update above.

3. Consider a Gaussian distribution $\mathcal{N}_4(0, \Sigma)$ with $K = \Sigma^{-1}$ satisfying the conditional independence restrictions of the graph $\mathcal{G} = (V, E)$ with $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 2\}, \{2, 3\}\{1, 4\}\{3, 4\}\}$.

   (a) Find two equations of degree 3 in σ_{13} and σ_{24} expressing these in terms of the variances $\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{44}$ and the covariances $\sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}$;
   Hint: Express the appropriate inverse element of the covariance matrix as a cofactor;
   The model is determined by $k_{13} = k_{24} = 0$. We let $x = \sigma_{13}$ and $y = \sigma_{24}$. By scaling of the data we might w.l.o.g. assume that $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{44} = 1$. Since Σ is assumed regular we must have $\det \Sigma \neq 0$ which in particular implies $|x| < 1$, $|y| < 1$, since x and y are correlations. We then get

$$k_{13} = \det \begin{pmatrix} \sigma_{21} & 1 & y \\ x & \sigma_{32} & \sigma_{34} \\ \sigma_{41} & y & 1 \end{pmatrix} / \det(\Sigma)$$

   so $k_{13} = 0$ if and only if the numerator is zero. This yields the equation

$$\sigma_{32}\sigma_{21} + \sigma_{34}\sigma_{41} + xy^2 = (\sigma_{32}\sigma_{41} + \sigma_{21}\sigma_{34})y + x.$$

   Similarly, $k_{24} = 0$ yields

$$\sigma_{41}\sigma_{21} + \sigma_{34}\sigma_{32} + x^2y = (\sigma_{41}\sigma_{32} + \sigma_{21}\sigma_{34})x + y.$$

   (b) Consider the likelihood equations based on observing a Wishart matrix $W = w$ with $W \sim \mathcal{W}(n, \Sigma)$. Use the answer under (a) to establish an equation of degree 5 for the maximum likelihood estimate of σ_{13}.
   Solving the second equation for y and exploiting $x^2 < 1$ yields

$$y = \frac{\sigma_{41}\sigma_{21} + \sigma_{34}\sigma_{32} - (\sigma_{41}\sigma_{32} + \sigma_{21}\sigma_{34})x}{1 - x^2} = \frac{a + bx}{1 - x^2}.$$

   Inserting this for y in the first equation and multiplying both sides with $(1 - x^2)^2$ yields an equation of degree 5.

(c) Assume next that $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{44} = 1$ and $\sigma_{12} = \sigma_{23} = \sigma_{34} = \rho$ and $\sigma_{14} = -\rho$. Show that then $\rho^2 < 1/2$.

The equations above simplify to $xy^2 = x$ and $yx^2 = y$. The only feasible solutions to these equations are $x = y = 0$ since $x = 1$ or $y = 1$ would imply that the covariance matrix becomes singular. The full covariance matrix therefore looks like

$$\Sigma = \begin{pmatrix} 1 & \rho & 0 & -\rho \\ \rho & 1 & \rho & 0 \\ 0 & \rho & 1 & \rho \\ -\rho & 0 & \rho & 1 \end{pmatrix}.$$

This matrix is positive definite if and only if all subdeterminants along the main diagonal are positive. We get

$$\det \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} = 1-\rho^2; \quad \det \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{pmatrix} = (1-2\rho^2); \quad \det \Sigma = (1-2\rho^2)^2.$$

These are all strictly positive if and only if $\rho^2 < 1/2$.