1. Let X_1, X_2, X_3, X_4, X_5 be independent with $X_i \sim \mathcal{N}(0,1)$. Define recursively

   $$Y_1 \leftarrow X_1,\ Y_2 \leftarrow X_2 + Y_1,\ Y_3 \leftarrow 2X_3 + Y_2,\ Y_4 \leftarrow X_4 + Y_3,\ Y_5 \leftarrow X_5 + 2Y_4.$$

   (a) Find the covariance matrix Σ of Y;

   (b) Find the concentration matrix $K = \Sigma^{-1}$ of Y.

   (c) Find a directed acyclic graph $\mathcal{D}$ so that Y is directed Markov w.r.t. $\mathcal{D}$.

   (d) Construct the dependence graph of Y;

   (e) Find the conditional distribution of Y_3 given $Y_1 = y_1, Y_2 = y_2, Y_4 = y_4, Y_5 = y_5$.

2. Consider a Gaussian distribution $\mathcal{N}_4(0,\Sigma)$ with $K = \Sigma^{-1}$ satisfying the conditional independence restrictions of the graph $\mathcal{G} = (V, E)$ with $V = \{1,2,3,4\}$ and $E = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\}$. Assume that the following Wishart matrix has been observed, with 10 degrees of freedom:

   $$\begin{pmatrix} 5 & 1 & 4 & 4 \\ 1 & 10 & 2 & 5 \\ 4 & 2 & 10 & 2 \\ 4 & 5 & 2 & 8 \end{pmatrix}.$$

   (a) Perform one full cycle of the IPS algorithm to find the MLE of the concentration matrix, starting with $K = I$.

   (b) Assume next that $K = \Sigma^{-1}$ satisfies the conditional independence restrictions of the graph $\mathcal{G}^* = (V, E^*)$ with $V = \{1,2,3,4\}$ and $E^* = \{\{1,2\},\{2,3\},\{3,4\}\}$. Find the maximum likelihood estimate of the concentration matrix.

3. Consider a Gaussian distribution $\mathcal{N}_4(0,\Sigma)$ with $K = \Sigma^{-1}$ satisfying the conditional independence restrictions of the graph $\mathcal{G} = (V, E)$ with $V = \{1,2,3,4\}$ and $E = \{\{1,2\},\{2,3\}\{1,4\}\{3,4\}\}$.

   (a) Find two equations of degree 3 in σ_{13} and σ_{24} expressing these in terms of the variances $\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{44}$ and the covariances $\sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}$;
   Hint: Express the appropriate inverse element of the covariance matrix as a cofactor;

   (b) Consider the likelihood equations based on observing a Wishart matrix $W = w$ with $W \sim \mathcal{W}(n, \Sigma)$. Use the answer under (a) to establish an equation of degree 5 for the maximum likelihood estimate of σ_{13}.

   (c) Assume next that $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{44} = 1$ and $\sigma_{12} = \sigma_{23} = \sigma_{34} = \rho$ and $\sigma_{14} = -\rho$. Show that then $\rho^2 < 1/2$.