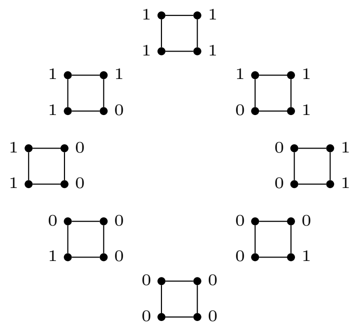
1. *You may find this question difficult! So skip initially if you struggle.* Consider the distribution over four binary variables which gives probability 1/8 to all of the 8 configurations displayed in the figure below:



Note that there are only four variables. The only reason that the graph (four-cycle) is repeated is to see the obvious pattern in the configuration.

Show that this distribution satisfies (G) with respect to the four cycle displayed, but the distribution does not factorize with respect to this graph, i.e., it does not satisfy (F).

Hint: Assume that it does factorize and show that if it is positive on these configurations, it must be positive on all 16 possible configurations of the four binary variables.

2. Consider the following generating class over $V = \{a, b, c, d, e, f, g, h\}$:

$$\mathcal{A} = \{\{a, c, e\}, \{b, c\}, \{b, h\}, \{d, e\}, \{d, f, h\}, \{d, g, h\}, \{f, g\}\}.$$

   (a) Find the dependence graph $G(\mathcal{A})$;

   (b) Find the factor graph $F(\mathcal{A})$;

   (c) Find the cliques of $G(\mathcal{A})$;

   (d) Show $\mathcal{A}$ is not conformal;

   (e) which of the following three statements are implied by $\mathcal{A}$?

$$\{a, b, c\} \perp\!\!\!\perp \{d, f, g\} \mid \{e, h\}, \quad \{a, b, c\} \perp\!\!\!\perp \{d, f, g, h\} \mid \{e\}, \quad d \perp\!\!\!\perp h \mid \{f, g\}.$$

3. Prove the *information inequality:* For non-negative numbers $a(x)_{x \in \mathcal{X}}$ and $b(x)_{x \in \mathcal{X}}$ with $\sum_x a(x) = \sum_x b(x)$ it holds that

$$\sum_{x \in \mathcal{X}} a(x) \log b(x) \leq \sum_{x \in \mathcal{X}} a(x) \log a(x)$$

where the inequality is strict unless $a(x) = b(x)$ for all $x \in \mathcal{X}$. For the expressions to make sense we use the convention that $0 \log 0 = 0$.

Hint: Show first the inequality $\log y \leq y - 1$.

4. The *entropy* $H(X)$ of a discrete random variable X is

$$H(X) = \mathbf{E}\{-\log f(X)\} = \sum_{x \in \mathcal{X}} -f(x) \log f(x).$$

   (a) Show that $H(X) \geq 0$;

   (b) Show that the entropy of the uniform distribution with $f(x) = 1/|\mathcal{X}|$ is $H(X) = \log |\mathcal{X}|$.

   (c) Show that the uniform distribution has maximal entropy, i.e. that

$$H(X) \leq \log |\mathcal{X}|.$$

5. Consider a three-way contingency table with variables A, B, C and cell counts

		C		
A	B	0	1	total
1	1	3	5	8
1	0	12	10	22
0	1	10	10	20
0	0	1	4	5

   Perform one cycle of the IPS algorithm for the hierarchical log-linear model with generator $\{\{A, B\}, \{B, C\}, \{A, C\}\}$.

6. Consider a generating class $\mathcal{A} = \{a, b\}$ with only two elements. Show that the IPS algorithm converges after a single cycle.