Wishart and Inverse Wishart Distributions

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BS2 Statistical Inference, Lecture 9, Hilary Term 2009

February 23, 2009
The Wishart distribution is the sampling distribution of the matrix of sums of squares and products. More precisely:

A random $d \times d$ matrix $W$ has a **d-dimensional Wishart distribution** with parameter $\Sigma$ and $n$ degrees of freedom if

$$W \overset{D}{=} \sum_{i=1}^{n} X_\nu X_\nu^\top$$

where $X_\nu \sim \mathcal{N}_d(0, \Sigma)$. We then write

$$W \sim \mathcal{W}_d(n, \Sigma).$$

The Wishart is the multivariate analogue to the $\chi^2$:

$$\mathcal{W}_1(n, \sigma^2) = \sigma^2 \chi^2(n).$$

If $W \sim \mathcal{W}_d(n, \Sigma)$ its mean is $E(W) = n\Sigma$. 
If $W_1$ and $W_2$ are independent with $W_i \sim \mathcal{W}_d(n_i, \Sigma)$, then

$$W_1 + W_2 \sim \mathcal{W}_d(n_1 + n_2, \Sigma).$$

If $A$ is an $r \times d$ matrix and $W \sim \mathcal{W}_d(n, \Sigma)$, then

$$AWA^\top \sim \mathcal{W}_r(n, A\Sigma A^\top).$$

For $r = 1$ we get that when $W \sim \mathcal{W}_d(n, \Sigma)$ and $\lambda \in \mathbb{R}^d$,

$$\lambda^\top W \lambda \sim \sigma_\lambda^2 \chi^2(n),$$

where $\sigma_\lambda^2 = \lambda^\top \Sigma \lambda$. 
If $W \sim \mathcal{W}_d(n, \Sigma)$, where $\Sigma$ is regular, then $W$ is regular with probability one if and only if $n \geq d$.

When $n \geq d$ the Wishart distribution has density

$$f_d(w \mid n, \Sigma) = c(d, n)^{-1}(\det \Sigma)^{-n/2}(\det w)^{(n-d-1)/2}e^{-tr(\Sigma^{-1}w)/2}$$

for $w$ positive definite, and 0 otherwise.

The Wishart constant $c(d, n)$ is

$$c(d, n) = 2^{nd/2}(2\pi)^{d(d-1)/4} \prod_{i=1}^{d} \Gamma\left\{(n + 1 - i)/2\right\}.$$
Let $W \sim \mathcal{W}_d(n, \Sigma)$ with $\Sigma$ regular and $n > d$. Then $W_{22}$ is regular with probability one and

(i) $W_{1|2}$ is independent of $(W_{12}, W_{22})$;

(ii) $W_{1|2} \sim \mathcal{W}_r(n - s, \Sigma_{1|2})$;

(iii) $W_{22} \sim \mathcal{W}_s(n, \Sigma_{22})$;

(iv) The conditional distribution of $W_{12}$ given $W_{22} = w_{22}$ is multivariate Gaussian $\mathcal{N}_{r \times s}(\Sigma_{12} \Sigma_{22}^{-1} w_{22}, \Lambda)$ where

$$\Lambda_{ij,kl} = \text{Cov}(W_{ij}, W_{kl} \mid W_{22} = w_{22}) = \sigma_{1|2}^{ij} w_{jl}.$$
In the special case with $\Sigma_{12} = 0$ this can be simplified to

$W_{1|2} \sim \mathcal{W}_r(n - s, \Sigma_{11})$ and

$W_{12} \mid W_{22} = w_{22} \sim \mathcal{N}_{r \times s}(0, \Lambda)$

with $\Lambda_{ij,kl} = \sigma_{ik}w_{jl}$.

It follows that in this case, i.e. when $\Sigma_{12} = 0$, it holds that

$W_{12}W_{22}^{-1}W_{21} \sim \mathcal{W}_r(s, \Sigma_{11})$. 
Consider $\mathcal{N}_3(0, \Sigma)$ with covariance matrix

$$\Sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$ 

The conditional distribution of $(X_1, X_2)$ given $X_3$ has covariance matrix

$$\Sigma_{12|3} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}.$$
Suppose we have $W \sim \mathcal{W}(n, \Sigma)$ with $\Sigma$ as specified. Then

$$W_{12|3} = \begin{pmatrix} W_{11} - W_{33}^{-1}W_{13}^2 & W_{12} - W_{33}^{-1}W_{13}W_{23} \\ W_{21} - W_{33}^{-1}W_{21}W_{23} & W_{22} - W_{33}^{-1}W_{23}^2 \end{pmatrix}$$

$$\sim \mathcal{W}(n-1, \Sigma_{12|3})$$

and independent of $(W_{13}, W_{23}, W_{33})$.

The conditional distribution of $(W_{13}, W_{23})^\top$ given $W_{33} = w_{33}$ is bivariate Gaussian, with mean

$$\left( \begin{array}{c} 1 \\ 1 \end{array} \right) \sigma_{33}^{-1} w_{33} = \left( \begin{array}{c} w_{33}/2 \\ w_{33}/2 \end{array} \right)$$

and covariance matrix

$$w_{33} \Sigma_{12|3} = \frac{w_{33}}{2} \left( \begin{array}{cc} 1 & 1 \\ 1 & 3 \end{array} \right).$$
If $W_1 \sim \mathcal{W}_d(f_1, \Sigma)$ and $W_2 \sim \mathcal{W}_d(f_2, \Sigma)$ with $f_1 \geq d$, then the distribution of

$$\Lambda = \frac{\det(W_1)}{\det(W_1 + W_2)}$$

is Wilks’ distribution and denoted by $\Lambda(d, f_1, f_2)$. It holds that

$$\Lambda \overset{D}{=} \prod_{i=1}^{d} B_i$$

where $B_i$ are independent and follow Beta distributions with

$$B_i \sim B\{(f_1 + 1 - i)/2, f_2/2\}.$$
Wilks’ distribution occurs as the likelihood ratio test for independence. Consider \( \mathcal{W} \sim \mathcal{W}_d(f, \Sigma) \) and the hypothesis that \( \Sigma_{12} = 0 \) for a fixed block partitioning of \( \Sigma \) into \( r \times r, r \times s \) and \( s \times s \) matrices. The likelihood ratio statistic then becomes

\[
\frac{L(\hat{K}_{11}, \hat{K}_{22})}{L(\hat{K})} = \left\{ \frac{\det(\mathcal{W})}{\det(\mathcal{W}_{11}) \det(\mathcal{W}_{22})} \right\}^{n/2} = U^{n/2},
\]

where

\[
U \sim \Lambda(r, f - s, s) = \Lambda(s, f - r, r).
\]

It follows that

\[
\Lambda(d, f_1, f_2) = \Lambda(f_2, f_1 + f_2 - d, d).
\]
Example: the bivariate case

Consider $Z = (X, Y)^\top$ and assume $Z \sim \mathcal{N}(0, \Sigma)$ with

$$\Sigma = \begin{pmatrix} \sigma_X^2 & \rho \sigma_X \sigma_Y \\ \rho \sigma_X \sigma_Y & \sigma_Y^2 \end{pmatrix}.$$ 

From data $Z_1, \ldots, Z_n$, form the Wishart matrix

$$W = \begin{pmatrix} \sum_i X_i^2 & \sum_i X_i Y_i \\ \sum_i X_i Y_i & \sum_i Y_i^2 \end{pmatrix}.$$ 

Wilks' $\Lambda$ for independence then becomes

$$\Lambda = LR^2/n = \frac{\sum_i X_i^2 \sum_i Y_i^2 - (\sum_i X_i Y_i)^2}{\sum_i X_i^2 \sum_i Y_i^2} = 1 - R^2.$$ 

This is $\Lambda(1, n - 1, 1)$ so $(n - 1)R^2/(1 - R^2) \sim F(n - 1, 1)$. 

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Hotelling’s $T^2$ is the equivalent of Student’s $t$-distribution. Let $Y \sim \mathcal{N}_d(\mu, c\Sigma)$, $W \sim \mathcal{W}_d(f, \Sigma)$ with $f \geq d$, and $Y \perp \perp W$.

$$T^2 = f(Y - \mu)^\top W^{-1}(Y - \mu)/c$$

is known as Hotelling’s $T^2$.

It holds that

$$\frac{1}{1 + T^2/f} \sim \Lambda(d, f, 1) = \Lambda(1, f - d + 1, d)$$

and

$$\frac{f - d + 1}{fd} T^2 \sim F(d, f + 1 - d)$$

where $F$ denotes Fisher’s $F$-distribution.
Recall that the Wishart density has the form

\[ f_d(w \mid n, \Sigma) \propto (\det w)^{(n-d-1)/2} e^{-\text{tr}(\Sigma^{-1}w)/2}. \]

Since the likelihood function for \( \Sigma \) is

\[ L(K) = (\det K)^{n/2} e^{-\text{tr}(KW)/2}, \]

a conjugate family of distributions for \( K \) is given by

\[ \pi(K; a, \Psi) \propto (\det K)^{a/2-1} e^{-\text{tr}(K\Psi)/2}, \]

which thus specifies a Wishart distribution for the concentration matrix.
We then say that $\Sigma$ follows an inverse Wishart distribution if $K = \Sigma^{-1}$ follows a Wishart distribution, formally expressed as

$$\Sigma \sim \mathcal{IW}_d(\delta, \Psi) \iff K = \Sigma^{-1} \sim \mathcal{W}_d(\delta + d - 1, \Psi^{-1})$$

i.e. if the density of $K$ has the form

$$f(K | \delta, \Psi) \propto (\det K)^{\delta/2 - 1} e^{-\text{tr}(\Psi K)/2}.$$

We repeat the expression for the standard Wishart density:

$$f_d(w | n, \Sigma) \propto (\det w)^{(n-d-1)/2} e^{-\text{tr}(\Sigma^{-1} w)/2}.$$ 

It follows that the family of inverse Wishart distributions is a conjugate family for $\Sigma$. 

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If the prior distribution of $\Sigma$ is $I\mathcal{W}_d(\delta, \Psi)$ and $W | \Sigma \sim \mathcal{W}_d(n, \Sigma)$, we get for the posterior density of $K$ that

$$f(K | \delta, \Psi, W) \propto (\det K)^{n/2} e^{-\text{tr}(KW)/2} \times (\det K)^{\delta/2-1} e^{-\text{tr}(\Psi K)/2} = (\det K)^{(n+\delta)/2-1} e^{-\text{tr}(\Psi+W)K}/2,$$

and hence the posterior distribution is simply $I\mathcal{W}_d(\delta + n, \Psi + W) = I\mathcal{W}_d(\delta^*, \Psi^*)$.

We can thus interpret the parameter $\delta$ as a prior equivalent sample size and $\Psi$ as the value of a matrix of sums and squares and products from a previous sample.