Modularity on Random Graphs, Lattices and Embedded Graphs

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Modularity

- First introduced by Newman and Girvan 2004 as a measure of how well a network is clustered into communities.
- Many clustering algorithms; based on optimising modularity; including protein discovery and social networks.
- Finding the optimal partition of a graph shown to be NP-hard by Brandes et. al. 2007.


**Figure 1:** Modularity used to study effect of schizophrenia on brain cell interaction.
Let $G$ be a graph on $m$ edges and $\mathcal{A}$ a vertex partition of $V(G)$

**Modularity**
\[
q_{\mathcal{A}}(G) := \sum_{A \in \mathcal{A}} \left( \frac{e(A)}{m} - \left( \frac{\text{degsum}(A)}{2m} \right)^2 \right)
\]

**Max. Modularity**
\[
q(G) := \max_{\mathcal{A}} q_{\mathcal{A}}(G)
\]
**Definition**

Let $G$ be a graph on $m$ edges and $\mathcal{A}$ a vertex partition of $V(G)$

**Modularity**  \[ q_\mathcal{A}(G) := \sum_{A \in \mathcal{A}} \left( \frac{e(A)}{m} - \left( \frac{\text{degsum}(A)}{2m} \right)^2 \right) \]

**Max. Modularity**  \[ q(G) := \max_{\mathcal{A}} q_\mathcal{A}(G) \]

Notice the sum naturally splits into two components.

**Edge contribution**  \[ q^E_\mathcal{A}(G) := \sum_{A \in \mathcal{A}} \frac{e(A)}{m} \]

**Degree tax**  \[ q^D_\mathcal{A}(G) := \sum_{A \in \mathcal{A}} \left( \frac{\text{degsum}(A)}{2m} \right)^2 \]
**Edge contribution**

\[ q^E_G(A) := \sum_{A \in \mathcal{A}} \frac{e(A)}{m} \]

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**Example Graph**

![Example Graph Image]
Edge contribution

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3 Possible Partitions
**Edge contribution**

\[ q^E_{A}(G) := \sum_{A \in \mathcal{A}} \frac{e(A)}{m} \]

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\[ q^D_{A}(G) := \sum_{A \in \mathcal{A}} \left( \frac{\text{degsum}(A)}{2m} \right)^2 \]

**3 Possible Partitions**

- \( q^E_{A_1} = 0.96, \quad q^D_{A_1} = 0.56 \)
  \( q_{A_1} = 0.40 \)

- \( q^E_{A_2} = 0.94, \quad q^D_{A_2} = 0.50 \)
  \( q_{A_2} = 0.44 \)

- \( q^E_{A_3} = 0.59, \quad q^D_{A_3} = 0.29 \)
  \( q_{A_3} = 0.30 \)
**Theorem (McDiarmid, S.)**

Let $G_r$ be an $r$-regular random graph. Then with high probability -

<table>
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**Random $r$-Regular Graphs**

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**Lower Bounds $r = 3, \ldots, 8$**

Hamilton cycle construction, $\sqrt{n}$ parts.

**Lower Bounds $r = 9, 10$**

Two equal sized parts.
**Random $r$-Regular Graphs**

**Upper Bounds**

$$i_u(G) := \min_{|U| \leq un} \frac{1}{|U|} e(U, V \setminus U)$$

**edge expansion of small sets**

**Theorem (McDiarmid, S.)**

Let $G$ be an $r$-regular graph. Suppose for all $u \leq 1/2$ that

$$u + i_u(G)/r \geq \alpha.$$  

Then,

$$q(G) \leq \max\{1 - \alpha, 3/4\}.$$  

Results of Kolesnik and Wormald\(^1\) give numerical bounds on edge
expansion of small sets in random regular graphs whp.

\(^1\)B. Kolesnik and N. Wormald, Lower bounds for the isoperimetric numbers of random regular graphs, SIAM Journal on Discrete Mathematics 28, 553 (2014)
# Simulations

## Modularity of Random $r$-Regular Graphs (whp)

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   MATLAB, 10 000 nodes, reject if not simple graph.
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1. Graphs generated via configuration model
   MATLAB, 10 000 nodes, reject if not simple graph.
2. Modularity estimated via Louvain method
   Etienne Lefebvre 2007,
   MATLAB implementation by Antoine Scherrer ENS Lyon.
   (available from Vincent Blondel’s website)
   results averaged over 10 trials.
Modularity of Random $r$-Regular Graphs (whp)

$X$ - results of simulations

$q(G_r)$

$s(G_r^*)$

$r$

References


Phase Transition in Erdős-Rényi

Connected components in the random graph $G_{n,c/n}$.

- $c < 1$:
  - $O(\log n)$

- $c = 1$:
  - $\sim n^{2/3}$

- $c > 1$:
  - $\sim n$

[Image Credit: M. Kang, TU Graz.]
**Phase Transition in Erdős-Rényi**

**Connected components in the random graph $G_{n,c/n}$.**

- **$c < 1$**  
  $O(\log n)$

- **$c = 1$**  
  $\sim n^{2/3}$

- **$c > 1$**  
  $\sim n$

$q(G_{n,c/n}) \to 1$

$q(G_{n,c/n}) \not\to 1$

[Image Credit: M. Kang, TU Graz.]
**Theorem (R. Guimerà et. al.°)**

*Fix $d, z \in \mathbb{N}^+$ and let $\mathcal{R}$ be an $n$-vertex complete rectangular section of $\mathbb{Z}^d_z$. Then $q(\mathcal{R}) \geq 1 - (d + 1) \left( \frac{z+1}{2d} \right)^{\frac{d}{d+1}} \frac{1}{n^{\frac{1}{d+1}}}$*

**Definition** $\mathbb{Z}^d_z$: $d$-dim lattice with axis-parallel edges lengths $1, ..., z$.

**Figure 2:** Rectangular sections of $\mathbb{Z}^2_2$ (left) and $\mathbb{Z}^2_3$ (right).

---

**Theorem (R. Guimerà et. al.³)**

Fix $d, z \in \mathbb{N}^+$ and let $R$ be an $n$-vertex complete rectangular section of $\mathbb{Z}_z^d$. Then $q(R) \geq 1 - (d + 1) \left(\frac{z+1}{2d}\right)^{\frac{d}{d+1}} n^{-\frac{1}{d+1}} = 1 - \Theta(m^{-\frac{1}{d+1}})$

**Definition** $\mathbb{Z}_z^d$: $d$-dim lattice with axis-parallel edges lengths $1, \ldots, z$.

**Figure 2:** Rectangular sections of $\mathbb{Z}_2^2$ (left) and $\mathbb{Z}_3^2$ (right).

---

We extend this result to include any subgraph of the lattice $\mathbb{Z}_z^d$.

**Theorem (McDiarmid, S.)**

Fix $d, z \in \mathbb{N}^+$, and let $\mathcal{L}$ be an $m$-edge subgraph of $\mathbb{Z}_z^d$. Then

$$q(\mathcal{L}) = 1 - O\left(m^{-\frac{1}{d+1}}\right) \quad \text{as } m \to \infty.$$
An embedding $\alpha$ of a graph $G$ into $\mathbb{R}^d$ is said to have warp $\ell$ if

$$\forall x, y \in V(G), 1 \leq |\alpha(x) - \alpha(y)|$$

$$\forall uv \in E(G), |\alpha(u) - \alpha(v)| \leq \ell$$

**Theorem (McDiarmid, S.)**

Let $G$ be a graph, $d \geq 2$. Suppose $\alpha : V(G) \to \mathbb{R}^d$ embeds $G$ with warp $\ell$. Then $q(G) \geq 1 - O(\ell^{d-1} m^{-1} d+1)$. 
**Idea of Proof**

**Assumptions:** graph $G$ and mapping $\alpha : G \to \mathbb{R}^d$ such that

- $\forall x, y \in V(G), 1 \leq |\alpha(x) - \alpha(y)|$  \hspace{1cm} \text{min vertex separation}
- $\forall uv \in E(G), |\alpha(u) - \alpha(v)| \leq \ell$  \hspace{1cm} \text{max edge length}

These imply: $\Delta(G) \leq U(B_{2\ell})$  \#unit spheres in ball of radius $2\ell$.

Let $\mathcal{H}_s$ be a hypercube of side length $s$. Then the max sum of degrees of vertices embedded inside of any hypercube $\mathcal{H}_s$ is.

$$\text{degsum}_{\alpha(G)}(\mathcal{H}_s) \leq \Delta(G)U(\mathcal{H}_s)$$  \#unit spheres in hypercube.
**Idea of Proof**

**Assumptions:** graph $G$ and mapping $\alpha : G \rightarrow \mathbb{R}^d$ such that

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$\degsum_{\alpha(G)}(\mathcal{H}_s) \leq \Delta(G)U(\mathcal{H}_s)$ \hspace{1cm} \#unit spheres in hypercube.

**Lemma (McDiarmid, S.)**

Let $G$ be a graph. Suppose $\alpha : V(G) \rightarrow \mathbb{R}^d$ embeds $G$ with max edge length $\ell$. Then for $s \gg \ell$;

$$q(G) \geq 1 - \frac{\ell\sqrt{d}}{s} - \frac{\degsum(\mathcal{H}_s)}{2m}.$$
Trees of bounded degree, Bagrow 2012.
Trees with degree $o(n^{1/5})$, Montgolfier et. al. 2011.

**Theorem (McDiarmid, S.)**

Let $G$ be a graph with $m$ edges, treewidth $tw(G) = t$ and maximum degree $\Delta = \Delta(G)$. Then the modularity $q(G)$ satisfies

$$q(G) \geq 1 - 2((t + 1)\Delta / m)^{1/2}.$$  

For $m = 1, 2, \ldots$ let $G_m$ be a graph with $m$ edges. If $tw(G_m) \cdot \Delta(G_m) = o(m)$ then $q(G_m) \to 1$ as $m \to \infty$. 
Random Graphs on Surfaces via Treewidth

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**Corollary**

Fix a surface $S$ and let $G_S(n)$ be chosen uniformly from all graphs on $n$ vertices which embed into $S$ with no crossing edges. Then with high probability $q(G_S(n)) \geq 1 - O(ln n/\sqrt{n})$. 
OPEN QUESTIONS

Modularity =
Edge contribution

\[ q^E_A(G) := \sum_{A \in \mathcal{A}} \frac{e(A)}{m} \]

- Degree tax

\[ q^D_A(G) := \sum_{A \in \mathcal{A}} \left( \frac{\text{degsum}(A)}{2m} \right)^2 \]

1. Edge expansion of small sets

Is there a cubic graph \( G \) for which \( i_u(G) \geq 1, \forall u \)?

\[ \text{October 12, 2014} \]
**Open Questions**

**Modularity =**

**Edge contribution**

\[ q^E_A(G) := \sum_{A \in A} \frac{e(A)}{m} \]

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1. **Edge expansion of small sets**

Is there a cubic graph \( G \) for which \( i_u(G) \geq 1, \forall u \)?

2. **Improve bounds in random cubic.**

Let \( G_3 \) be a random cubic graph. Then whp,

\[ 0.66 \leq q(G_3) \leq 0.8 \]

Is the lower bound optimal? i.e. \( q(G_3) = 2/3 \) whp?

Construction based on finding a Hamilton cycle, then cutting into strips of \( \sqrt{n} \) vertices.
**Edge expansion**

\[ i(G) := \min_{|U| \leq n/2} \frac{1}{|U|} e(U, V \setminus U) \]

**Small sets modularity**

\[ q_\delta(G) := \max_{A : |A| < \delta n, \forall A \in \mathcal{A}} q_A(G) \]

**Theorem (McDiarmid, S.)**

For any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that the following holds. Let \( r \geq 3 \) and let \( G \) be an \( r \)-regular graph with at least \( \delta^{-1} \) vertices. Then,

\[ q_\delta(G) < 1 - \frac{2}{r} i(G) + \varepsilon. \]

**Observation**

\( q(G) \leq \max\{1 - \frac{1}{r} i(G), \frac{3}{4}\} \).

Why? Fix \( \mathcal{A} = \{A_1, \ldots, A_k\} \).

(a) If some \( |A_i| > n/2 \) then degree tax is at least \( (|A_i| r / r n)^2 > \frac{1}{4} \).

(b) If all \( |A_i| \leq n/2 \) use edge expansion. The number of edges between parts is \( \frac{1}{2} \sum_i e(A_i, V \setminus A_i) \geq \frac{1}{2} |A_i| i(G) = \frac{1}{2} i(G) n. \)

So the edge contribution is less than \( 1 - \frac{2}{rn} \frac{1}{2} i(G) n = 1 - \frac{1}{r} i(G) \).
**Lemma (McDiarmid, S.)**

Fix vertices $[n]$ with weights $w(1) \geq \ldots \geq w(n) \geq 0$. Let $M$ be a perfect matching such that $|a - b| \leq t$ for all edges $ab \in M$. For any red-blue colouring of $M$,

$$\left| \sum_{v \text{ red}} w(v) - \sum_{u \text{ blue}} w(u) \right| \leq t (w(1) - w(n)).$$
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*For any red-blue colouring of \(\mathcal{M}\),*

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$t = 1$

$$\left| \sum_{v \, \text{red}} w(v) - \sum_{u \, \text{blue}} w(u) \right| = w(1) - w(2) + w(3) - \ldots + w(7) - w(8)$$
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\(t = 3\)
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Fix vertices \([n]\) with weights \(w(1) \geq \ldots \geq w(n) \geq 0\). Let \(\mathcal{M}\) be a perfect matching such that \(|a - b| \leq t\) \(\forall\) edges \(ab \in \mathcal{M}\).

For any red-blue colouring of \(\mathcal{M}\),

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\(t = 3\)
**Proof Outline**

(RTP: $q_A(G) \leq 1 - \frac{2}{r} i(G) + \varepsilon$).

Fix $\varepsilon$ and a partition $A = A_1, \ldots, A_k$ where $\delta n > |A_1| \geq \ldots \geq |A_k|$.
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**Step 1.** Choose $t$, and factor $K_{t+1}$ into perfect matchings.

These induce pairings on parts in $G$. 

$G : A_1, \ldots, A_k$
Fix $\varepsilon$ and a partition $\mathcal{A} = A_1, \ldots, A_k$ where $\delta n > |A_1| \geq \ldots \geq |A_k|$.

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(RTP: $q_{\mathcal{A}}(G) \leq 1 - \frac{2}{r} i(G) + \varepsilon$).
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**Step 1.** Choose $t$, and factor $K_{t+1}$ into perfect matchings. These induce pairing on parts of $G$. 

$G : A_1, \ldots, A_k$
Fix $\varepsilon$ and a partition $\mathcal{A} = A_1, \ldots, A_k$ where $\delta n > |A_1| \geq \ldots \geq |A_k|$.

**Step 1.** Choose $t$, and factor $K_{t+1}$ into perfect matchings. Choose $P_\alpha$ to minimise edges between paired parts in $G$. 

(RTP: $q_{\mathcal{A}}(G) \leq 1 - \frac{2}{r} i(G) + \varepsilon$).
Proof Outline

(RTP: $q_A(G) \leq 1 - \frac{2}{r} i(G) + \varepsilon$).

Fix $\varepsilon$ and a partition $A = A_1, \ldots, A_k$ where $\delta n > |A_1| \geq \ldots \geq |A_k|$.

**Step 1.** Choose $t$, and factor $K_{t+1}$ into perfect matchings.

Choose $P_\alpha$ to minimise edges between paired parts in $G$.

$\therefore E^\alpha_{PAIRS} := \# \text{ edges between paired parts} \leq m/t.$
**Proof Outline**

(RTP: $q_A(G) \leq 1 - \frac{2}{r} i(G) + \varepsilon$).

Fix $\varepsilon$ and a partition $\mathcal{A} = A_1, \ldots, A_k$ where $\delta n > |A_1| \geq \ldots \geq |A_k|$.

**Step 1.** Choose $t$, and factor $K_{t+1}$ into perfect matchings.
Choose $P_\alpha$ to minimise edges between paired parts in $G$.

\[ \therefore E^\alpha_{PAIRS} := \# \text{ edges between paired parts} \leq m/t. \]

\[ E^\alpha_{\neg PAIRS} := \# \text{ edges between distinct non-paired parts}. \]
**Proof Outline**

(RTP: \( q_A(G) \leq 1 - \frac{2}{r} i(G) + \varepsilon \)).

Fix \( \varepsilon \) and a partition \( A = A_1, \ldots, A_k \) where \( \delta n > |A_1| \geq \ldots \geq |A_k| \).

**Step 1.** Choose \( t \), and factor \( K_{t+1} \) into perfect matchings. Choose \( P_\alpha \) to minimise edges between paired parts in \( G \).

\[ \therefore E_{\alpha \text{PAIRS}} := \# \text{ edges between paired parts} \leq \frac{m}{t}. \]

\[ E_{\alpha \neg \text{PAIRS}} := \# \text{ edges between distinct non-paired parts}. \]

**Step 2.** For each pair in \( P_\alpha \) randomly colour parts red and blue. For each part not in \( P_\alpha \) randomly colour it red or blue.
**Proof Outline**

(RTP: \( q_{\mathcal{A}}(G) \leq 1 - \frac{2}{r} i(G) + \varepsilon \)).

Fix \( \varepsilon \) and a partition \( \mathcal{A} = A_1, \ldots, A_k \) where \( \delta n > |A_1| \geq \ldots \geq |A_k| \).

**Step 1.** Choose \( t \), and factor \( K_{t+1} \) into perfect matchings.

Choose \( P_\alpha \) to minimise edges between paired parts in \( G \).

\[ \therefore E_{\text{PAIRS}}^\alpha := \# \text{ edges between paired parts} \leq m/t. \]

\[ E_{\neg \text{PAIRS}}^\alpha := \# \text{ edges between distinct non-paired parts}. \]

**Step 2.** For each pair in \( P_\alpha \) randomly colour parts red and blue.

For each part not in \( P_\alpha \) randomly colour it red or blue.

\[ |\# \text{red} V - \# \text{blue} V| \]
Proof Outline

Fix $\varepsilon$ and a partition $\mathcal{A} = A_1, \ldots, A_k$ where $\delta n > |A_1| \geq \ldots \geq |A_k|$. 

**Step 1.** Choose $t$, and factor $K_{t+1}$ into perfect matchings. Choose $P_\alpha$ to minimise edges between paired parts in $G$. 

\[ E^\alpha_{PAIRS} := \# \text{ edges between paired parts} \leq m/t. \]

\[ E^\alpha_{\neg PAIRS} := \# \text{ edges between distinct non-paired parts}. \]

**Step 2.** For each pair in $P_\alpha$ randomly colour parts red and blue. For each part not in $P_\alpha$ randomly colour it red or blue. 

\[ |\# \text{red} V - \# \text{blue} V| \leq t(|A_1| - |A_j|) - t|A_{j+1}| \leq t|A_1| \text{ by Lemma} \]
Fix $\varepsilon$ and a partition $\mathcal{A} = A_1, \ldots, A_k$ where $\delta n > |A_1| \geq \ldots \geq |A_k|$. 

**Step 1.** Choose $t$, and factor $K_{t+1}$ into perfect matchings. Choose $P_\alpha$ to minimise edges between paired parts in $G$.

\[ E_{\alpha}^{\text{PAIRS}} := \# \text{ edges between paired parts} \leq \frac{m}{t}. \]

\[ E_{\alpha}^{\text{not PAIRS}} := \# \text{ edges between distinct non-paired parts}. \]

**Step 2.** For each pair in $P_\alpha$ randomly colour parts red and blue. For each part not in $P_\alpha$ randomly colour it red or blue.

\[ |\# \text{ red} V - \# \text{ blue} V| \leq t(|A_1| - |A_j|) - t|A_{j+1}| \leq t|A_1| \text{ by Lemma} \]

\[ E_{R,B}^{\alpha} := \# \text{ edges between red and blue parts}. \]

(RTP: $q_{\mathcal{A}}(G) \leq 1 - \frac{2r}{i(G)} + \varepsilon$).
Proof Outline

(RTP: $q_A(G) \leq 1 - \frac{2}{r}i(G) + \varepsilon$).

Fix $\varepsilon$ and a partition $A = A_1, \ldots, A_k$ where $\delta n > |A_1| \geq \ldots \geq |A_k|$.

**Step 1.** Choose $t$, and factor $K_{t+1}$ into perfect matchings.
Choose $P_\alpha$ to minimise edges between paired parts in $G$.

\[E^\alpha_{PAIRS} := \# \text{ edges between paired parts} \leq m/t.\]
\[E^\alpha_{\neg PAIRS} := \# \text{ edges between distinct non-paired parts}.\]

**Step 2.** For each pair in $P_\alpha$ randomly colour parts red and blue.
For each part not in $P_\alpha$ randomly colour red or blue.

$|\# red V - \# blue V| \leq t(|A_1| - |A_j|) - t|A_{j+1}| \leq t|A_1|$ by Lemma

$E^\alpha_{R,B} := \# \text{ edges between red and blue parts}$.

\[E^\alpha_{R,B} \geq i(G) \times \min\{\# red V, \# blue V\} \geq i(G)(\frac{n}{2} - \frac{t}{2}|A_1|)\]
Proof Outline

(RTP: $q_A(G) \leq 1 - \frac{2}{r} i(G) + \varepsilon$).

Fix $\varepsilon$ and a partition $A = A_1, \ldots, A_k$ where $\delta n > |A_1| \geq \ldots \geq |A_k|$.

**Step 1.** Choose $t$, and factor $K_{t+1}$ into perfect matchings.
Choose $P_\alpha$ to minimise edges between paired parts in $G$.

$$E_{\text{PAIRS}}^\alpha := \# \text{ edges between paired parts} \leq m / t.$$  
$$E_{\neg \text{PAIRS}}^\alpha := \# \text{ edges between distinct non-paired parts}.$$  

**Step 2.** For each pair in $P_\alpha$ randomly colour parts red and blue.
For each part not in $P_\alpha$ randomly colour red or blue.

$$|\# \text{red} V - \# \text{blue} V| \leq t(|A_1| - |A_j|) - t|A_{j+1}| \leq t|A_1|$$ by Lemma

$$E_{R,B}^\alpha := \# \text{ edges between red and blue parts}.$$  

$$E_{R,B}^\alpha \geq i(G) \times \min\{\# \text{red} V, \# \text{blue} V\} \geq i(G)(\frac{n}{2} - \frac{t}{2}|A_1|)$$

but, $\mathbb{E}[E_{R,B}^\alpha] = E_{\text{PAIRS}}^\alpha + \frac{1}{2} E_{\neg \text{PAIRS}}^\alpha$.  

$G: A_1, \ldots, A_k$
**Proof Outline**

(RTP: $q_A(G) \leq 1 - \frac{2}{r} i(G) + \varepsilon$).

Fix $\varepsilon$ and a partition $A = A_1, \ldots, A_k$ where $\delta n > |A_1| \geq \ldots \geq |A_k|$.

**Step 1.** Choose $t$, and factor $K_{t+1}$ into perfect matchings. Choose $P_\alpha$ to minimise edges between paired parts in $G$.

$E_{\alpha}^{\text{PAIRS}} := \#$ edges between paired parts $\leq m/t$.

$E_{\alpha}^{\neg \text{PAIRS}} := \#$ edges between distinct non-paired parts.

**Step 2.** For each pair in $P_\alpha$ randomly colour parts red and blue. For each part not in $P_\alpha$ randomly colour it red or blue.

$|\# \text{red} V - \# \text{blue} V| \leq t(|A_1| - |A_j|) - t|A_{j+1}| \leq t|A_1|$ by Lemma

$E_{R,B}^{\alpha} := \#$ edges between red and blue parts.

$\therefore E_{R,B}^{\alpha} \geq i(G) \times \min\{|\# \text{red} V, \# \text{blue} V\} \geq i(G)(\frac{n}{2} - \frac{t}{2}|A_1|)$

but, $\mathbb{E}[E_{R,B}^{\alpha}] = E_{\text{PAIRS}}^{\alpha} + \frac{1}{2} E_{\neg \text{PAIRS}}^{\alpha}$

**Finish.** We now have an upper bound for the edge contribution.
Fix $\varepsilon$ and a partition $\mathcal{A} = A_1, \ldots, A_k$ where $\delta n > |A_1| \geq \ldots \geq |A_k|$.  

**Step 1.** 

\[ \therefore E_{\text{PAIRS}}^\alpha := \# \text{ edges between paired parts} \leq m/t. \]
\[ E_{\neg \text{PAIRS}}^\alpha := \# \text{ edges between distinct non-paired parts}. \]

**Step 2.** 

\[ E_{R,B}^\alpha := \# \text{ edges between red and blue parts}. \]
\[ \therefore E_{R,B}^\alpha \geq i(G) \times \min\{|\text{red } V|, |\text{blue } V|\} \geq i(G)(\frac{n}{2} - \frac{t}{2}|A_1|) \]

but, $E[E_{R,B}^\alpha] = E_{\text{PAIRS}}^\alpha + \frac{1}{2} E_{\neg \text{PAIRS}}^\alpha$ 

**Finish.** We now have an upper bound for the edge contribution.
Proof Outline

(RTP: $q_A(G) \leq 1 - \frac{2}{r} i(G) + \varepsilon$).

Fix $\varepsilon$ and a partition $A = A_1, \ldots, A_k$ where $\delta n > |A_1| \geq \ldots \geq |A_k|$.

Step 1.

$\therefore E^{\alpha}_{PAIRS} := \#$ edges between paired parts $\leq m/t$.

$E^\alpha_{\neg PAIRS} := \#$ edges between distinct non-paired parts.

Step 2.

$E^\alpha_{R,B} := \#$ edges between red and blue parts.

$\therefore E^\alpha_{R,B} \geq i(G) \times \min\{\#red V, \#blue V\} \geq i(G)(\frac{n}{2} - \frac{t}{2}|A_1|)$

but, $\mathbb{E}[E^\alpha_{R,B}] = E^\alpha_{PAIRS} + \frac{1}{2} E^\alpha_{\neg PAIRS}$

Finish. We now have an upper bound for the edge contribution.

$q^E_A(G) = \frac{1}{m} \sum_{A \in A} E(A) = 1 - \frac{1}{m}(E^\alpha_{PAIRS} + E^\alpha_{\neg PAIRS})$
Proof Outline

(RTP: \( q_A(G) \leq 1 - \frac{2}{r} i(G) + \varepsilon \)).

Fix \( \varepsilon \) and a partition \( \mathcal{A} = A_1, \ldots, A_k \) where \( \delta n > |A_1| \geq \ldots \geq |A_k| \).

**Step 1.**

\[ \therefore E^\alpha_{\text{PAIRS}} := \# \text{ edges between paired parts} \leq \frac{m}{t}. \]

\[ E^\alpha_{\neg\text{PAIRS}} := \# \text{ edges between distinct non-paired parts}. \]

**Step 2.**

\[ E_{R,B}^\alpha := \# \text{ edges between red and blue parts}. \]

\[ \therefore E_{R,B}^\alpha \geq i(G) \times \min\{ \#\text{red } V, \#\text{blue } V \} \geq i(G)\left(\frac{n}{2} - \frac{t}{2}|A_1|\right) \]

but, \( \mathbb{E}[E_{R,B}^\alpha] = E^\alpha_{\text{PAIRS}} + \frac{1}{2} E^\alpha_{\neg\text{PAIRS}} \)

**Finish.** We now have an upper bound for the edge contribution.

\[ q^E_A(G) = \frac{1}{m} \sum_{A \in \mathcal{A}} E(A) = 1 - \frac{1}{m}(E^\alpha_{\text{PAIRS}} + E^\alpha_{\neg\text{PAIRS}}) \]

\[ = 1 - \frac{1}{m}(2\mathbb{E}[E_{R,B}^\alpha] - E^\alpha_{\text{PAIRS}}) \]
Fix $\varepsilon$ and a partition $\mathcal{A} = A_1, \ldots, A_k$ where $\delta n > |A_1| \geq \ldots \geq |A_k|$. 

**Step 1.**

\[ E^\alpha_{\text{PAIRS}} := \# \text{ edges between paired parts} \leq \frac{m}{t}.\]

\[ E^\alpha_{\neg \text{PAIRS}} := \# \text{ edges between distinct non-paired parts}.\]

**Step 2.**

\[ E^\alpha_{R,B} := \# \text{ edges between red and blue parts}.\]

\[ \therefore E^\alpha_{R,B} \geq i(G) \times \min\{\# \text{red} V, \# \text{blue} V\} \geq i(G)\left(\frac{n}{2} - \frac{t}{2}|A_1|\right) \]

but, $\mathbb{E}[E^\alpha_{R,B}] = E^\alpha_{\text{PAIRS}} + \frac{1}{2}E^\alpha_{\neg \text{PAIRS}}$

**Finish.** We now have an upper bound for the edge contribution.

\[ q^E_\mathcal{A}(G) = \frac{1}{m} \sum_{A \in \mathcal{A}} E(A) = 1 - \frac{1}{m}(E^\alpha_{\text{PAIRS}} + E^\alpha_{\neg \text{PAIRS}}) \]

\[ = 1 - \frac{1}{m}(2\mathbb{E}[E^\alpha_{R,B}] - E^\alpha_{\text{PAIRS}}) \geq 1 - \frac{2}{m}i(G)(n - t\delta) - \frac{1}{t} \]
Fix $\varepsilon$ and a partition $\mathcal{A} = A_1, \ldots, A_k$ where $\delta n > |A_1| \geq \ldots \geq |A_k|$. 

**Step 1.**

Choose $t$, and factor $K_t + 1$ into perfect matchings. 

$\therefore E^\alpha_{\text{PAIRS}} := \# \text{ edges between paired parts} \leq \frac{m}{t}$. 

$E^\alpha_{\text{\neg PAIRS}} := \# \text{ edges between distinct non-paired parts}$. 

**Step 2.**

$E^\alpha_{R,B} := \# \text{ edges between red and blue parts}$. 

$\therefore E^\alpha_{R,B} \geq i(G) \times \min\{\# \text{red} V, \# \text{blue} V\} \geq i(G)(\frac{n}{2} - \frac{t}{2}|A_1|)$

but, $\mathbb{E}[E^\alpha_{R,B}] = E^\alpha_{\text{PAIRS}} + \frac{1}{2} E^\alpha_{\text{\neg PAIRS}}$

**Finish.** We now have an upper bound for the edge contribution.

$q^E_{\mathcal{A}}(G) = \frac{1}{m} \sum_{A \in \mathcal{A}} E(A) = 1 - \frac{1}{m}(E^\alpha_{\text{PAIRS}} + E^\alpha_{\text{\neg PAIRS}})$

$= 1 - \frac{1}{m}(2\mathbb{E}[E^\alpha_{R,B}] - E^\alpha_{\text{PAIRS}}) \leq 1 - \frac{2}{r} i(G) - \frac{2t\delta}{r} - \frac{1}{t}$
Proof Outline

(RTP: \( q_A(G) \leq 1 - \frac{2}{r} i(G) + \varepsilon \)).

Fix \( \varepsilon \) and a partition \( A = A_1, \ldots, A_k \) where \( \delta n > |A_1| \geq \ldots \geq |A_k| \).

**Step 1.**

\[
\therefore E^\alpha_{PAIRS} := \# \text{ edges between paired parts} \leq \frac{m}{t}.
\]

\[
E^\alpha_{\neg PAIRS} := \# \text{ edges between distinct non-paired parts}.
\]

**Step 2.**

\[
E^\alpha_{R,B} := \# \text{ edges between red and blue parts}.
\]

\[
\therefore E^\alpha_{R,B} \geq i(G) \times \min\{\# \text{ red } V, \# \text{ blue } V\} \geq i(G)(\frac{n}{2} - \frac{t}{2}|A_1|)
\]

but, \( \mathbb{E}[E^\alpha_{R,B}] = E^\alpha_{PAIRS} + \frac{1}{2} E^\alpha_{\neg PAIRS} \)

**Finish.** We now have an upper bound for the edge contribution.

\[
q^E_A(G) = \frac{1}{m} \sum_{A \in A} E(A) = 1 - \frac{1}{m}(E^\alpha_{PAIRS} + E^\alpha_{\neg PAIRS})
\]

\[
= 1 - \frac{1}{m} \left( 2\mathbb{E}[E^\alpha_{R,B}] - E^\alpha_{PAIRS} \right) \leq 1 - \frac{2}{r} i(G) - \frac{2t\delta}{r} - \frac{1}{t}
\]

\( \therefore \) choose \( \delta \), \( t \) and we are done. \( \square \)
**Lemma (McDiarmid, S.)**

Fix vertices \([n]\) with weights \(w(1) \geq \ldots \geq w(n) \geq 0\). Let \(M\) be a perfect matching such that \(|a - b| \leq t\), \(\forall\) edges \(ab \in M\). For any red-blue colouring of \(M\),

\[
\left| \sum_{v \text{ red}} w(v) - \sum_{u \text{ blue}} w(u) \right| \leq t (w(1) - w(n)).
\]

**Theorem (McDiarmid, S.)**

For any \(\varepsilon > 0\) there exists \(\delta > 0\) such that the following holds. Let \(r \geq 3\) and let \(G\) be an \(r\)-regular graph with at least \(\delta^{-1}\) vertices. Then

\[
q_\delta(G) < 1 - \frac{2}{r} i(G) + \varepsilon.
\]