Liu A K, Belliveau J W, Dale A M 1998 Spatiotemporal imaging of human brain activity using functional MRI constrained magnetoencephalography data: Monte Carlo simulations. *Proceedings of the National Academy of Sciences of the United States of America* **95**: 8945–50

Moonen C T, Bandettini P A 1999 *Functional MRI*. Springer, New York

Ogawa S, Lee T M, Kay A R, Tank D W 1990 Brain magnetic resonance imaging with contrast depedent on blood oxygenation. *Proceedings of the National Academy of Sciences of the United States of America* **87**: 9868–72

Ogawa S, Tank D W, Menon R, Ellermann J M, Kim S-G, Merkle H, Ugurbil K 1992 Intrinsic signal changes accompanying sensory stimulation: Functional brain mapping with magnetic resonance imaging. *Proceedings of the National Academy of Sciences of the United States of America* **89**: 5951–5

Pauling L, Coryell C D 1936 The magnetic properties and structure of hemoglobin, oxyghemoglobin and caronmonoxyhemoglobin. *Proceedings of the National Academy of Sciences of the United States of America* **22**: 210–6

Petersen S E, Fox P T, Posner M I, Mintum M A, Raichle M E 1988 Positron emission tomographic studies of the cortical anatomy of single-word processing. *Nature* **331**: 585–9

Petersen S E, Fox P T, Posner M I, Mintun M A, Raichle M E 1989 Positron emission tomographic studies of the processing of single words. *Journal of Cognitive Neuroscience* **1**: 153–70

Phelps M E, Hoffman E J, Mullani N A, Ter-Pogossian M M 1975 Application of annihilation coincidence detection to transaxial reconstruction tomography. *Journal of Nuclear Medicine* **16**: 210–24

Raichle M E 1998 Behind the scenes of functional brain imaging: A historical and physiological perspective. *Proceedings of the National Academy of Sciences of the United States of America* **95**: 765–72

Raichle M E 2000 A brief history of human functional brain mapping. In: Toga A W, Mazziotta J C (eds.) *Brain Mapping. The Systems*. Academic Press, San Diego, pp. 33–75

Raichle M E, Fiez J A, Videen T O, MacLeod A M, Pardo J V, Fox P T, Petersen S E 1994 Practice-related changes in human brain functional anatomy during nonmotor learning. *Cerebral Cortex* **4**: 8–26

Raichle M E, MacLeod A M, Snyder A Z, Powers W J, Gusnard D A, Shulman G L 2001 A default mode of brain function. *Proceedings of the National Academy of Sciences of the United States of America* **98**: 676–82

Rosen B R, Buckner R L, Dale A M 1998 Event-related function MRI: Past, present and future. *Proceedings of the National Academy of Sciences of the United States of America* **95**: 773–80

Talairach J, Tournoux P 1988 *Coplanar Stereotaxic Atlas of the Human Brain*. Thieme Medical Publishers, New York

Ter-Pogossian M M, Phelps M E, Hoffman E J, Mullani N A 1975 A positron-emission tomograph for nuclear imaging (PETT). *Radiology* **114**: 89–98

Thulborn K R, Waterton J C, Matthews P M, Radda G K 1982 Oxygenation dependence of the transverse relaxation time of water protons in whole blood at high field. *Biochimica et Biophysica Acta* **714**: 265–70

Toga A W, Mazziotta J C 1996 *Brain Mapping. The Methods*. Academic Press, New York

Toga A W, Mazziotta J C 2000 *Brain Mapping. The Systems*. Academic Press, New York

Wagner A D, Schacter D L, Rotte M, Koutstaal W, Maril A, Dale A M, Rosen B R, Buckner R L 1998 Building memories: Remembering and forgetting of verbal experiences as predicted by brain activity. *Science* **281**: 1188–91

M. E. Raichle

Functional Data Analysis

Most statistical analyses involve one or more observations taken on each of a number of individuals in a sample, with the aim of making inferences about the general population from which the sample is drawn. In an increasing number of fields, these observations are curves or images. Curves and images are examples of functions, since an observed intensity is available at each point on a line segment, a portion of a plane, or a volume. For this reason, we call observed curves and images 'functional data,' and statistical methods for analyzing such data are described by the term 'functional data analysis' (FDA), coined by Ramsay and Dalzell (1991). Though the individual methods and techniques are important, functional data analysis is also a general way of thinking, where the basic unit of information is the entire observed function rather than a string of numbers. A few illustrations will be offered in this article, but Ramsay and Silverman (1997) may be consulted for many more examples and methods, for consideration of the underlying philosophy, and for further references to the literature.

The goals of functional data analysis are essentially the same as for other branches of statistics, and include the following:

(a) to represent and transform the data in ways that aid further analysis,

(b) to display the data so as to highlight various characteristics,

(c) to study important sources of pattern and variation among the data, and

(d) to explain variation in an outcome or dependent variable by using input or independent variable information.

We shall illustrate the nature of functional data, these goals, and FDA tools available or under development, through a series of examples.

1. Human Growth Data: Looking at Velocity and Acceleration

Figure 1 shows the growth patterns of 10 Swiss boys involved in the Zurich Longitudinal Growth Study (Falkner 1960). Each boy's height is measured at 29 separate ages, but it is more instructive to think of this as a sample of 10 growth curves or functions than as a set of 290 numerical observations. Of particular interest in the study of growth is the way that growth speeds up and slows down at various stages of

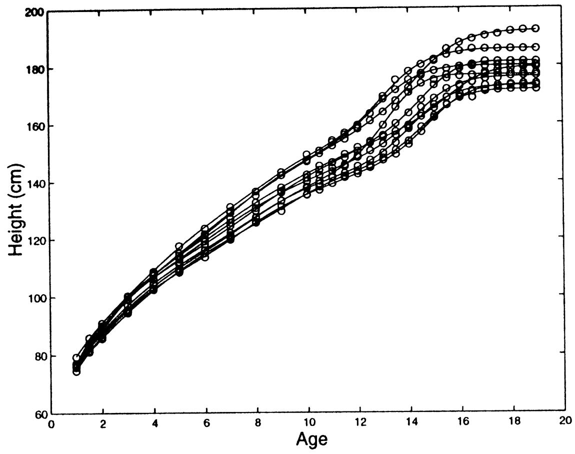

Figure 1
The heights of 10 Swiss boys measured at 29 ages
Notes: Points indicate unequally spaced ages of measurement; solid line is a function fitted to these data by monotone smoothing.

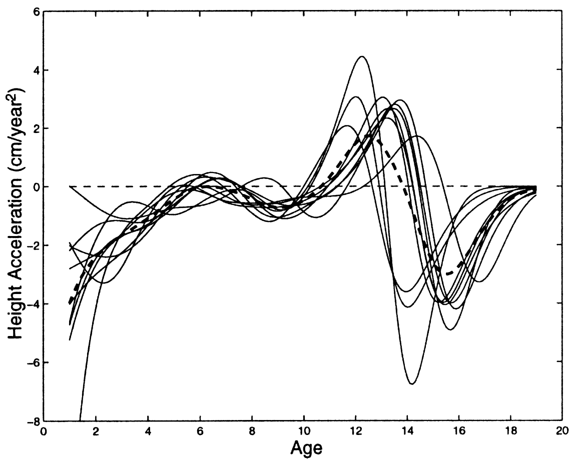

Figure 2
The estimated accelerations of height for 10 Swiss boys
Notes: Heavy dashed line is cross-sectional mean. This is a rather poor summary of the curves because of the presence of the timing of the pubertal growth spurt, called phase variation.

development, and this is much better illustrated by studying the velocity and acceleration of height. Figure 2 gives estimates of the acceleration or second derivative of these 10 height functions, and shows that around puberty there is a strong pulse of acceleration in growth, followed by sharp deceleration. This is the pubertal growth spurt (PGS), and there is also often a smaller spurt, hardly visible in Fig. 1 but obvious in Fig. 2, at around age 6.

2. *The Mean Function and the Registration Problem*

How would one estimate an overall acceleration curve to quantify the size of these spurts? The dashed curve in Fig.2 is the average acceleration over the sample at each time point, but the PGS for this average is smaller in size and longer in duration than that of any of the individual spurts. Averages are supposed to look like typical observations; so what went wrong?

The problem is that the curves vary in two ways, termed amplitude variation and phase variation. Alternatively, these are called vertical and horizontal variation, respectively, since amplitude concerns the height of a function at a given point, and phase the location in time of a feature such as an acceleration peak. In short, the actual acceleration peaks in the PGS vary in both size and timing. An artificial example demonstrating these two sorts of variation is presented in Fig. 3. The phase shift displayed in this figure is constant across the timescale, but many data sets require more complicated and nonlinear transformations of the time axis in order to align curve features like the peaks and valleys in Fig. 3. The presence of phase variation arises in FDA because many processes involve a system time, or in this case biological time, that does not unfold at the same rate as clock time. In this case, the biological clock of each boy marks off time at its own unique rate.

A key part of the FDA methodology is time-warping or registration of the data curves to align specific features or to minimize variability, since estimating the mean and most other analyses are thrown off by the presence of phase variation. The time-warping functions may be simple constants, linear functions, or more complicated, nonlinear

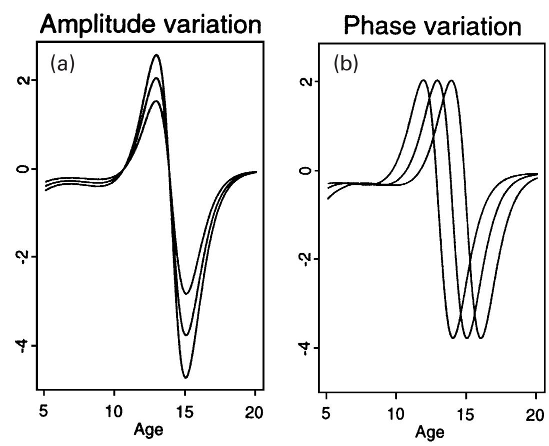

Figure 3
Panel (a): three height acceleration curves varying only in amplitude; Panel (b): three curves varying only in phase

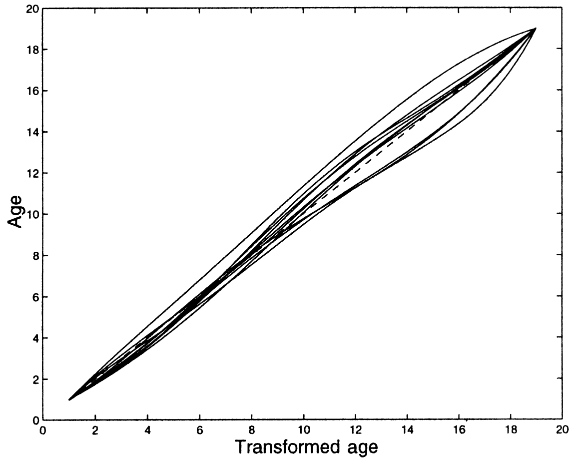

Figure 4
Time-warping functions taking chronological age on the horizontal axis into biological age on the vertical axis

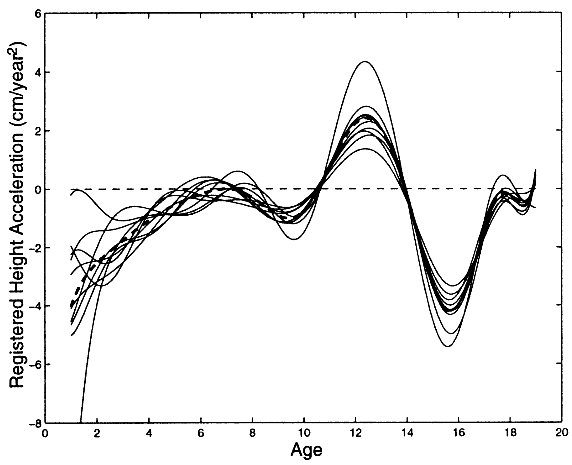

Figure 5
The height acceleration curves in Fig. 2 aligned or registered by measuring height with respect to the biological time shown in Fig. 4
Notes: Heavy dashed line is the mean acceleration. This provides a much better summary of the actual curves than Fig. 2

functions. The amount or shape of warp may be of great interest. See Ramsay and Silverman (1997, Chaps. 5 and 8). Figure 4 shows time-warping functions and registered height acceleration curves for the growth curves of Fig. 1 estimated using a method developed by Ramsay and Li (1998). The time-warping curve for each individual shows his maturation pattern. A warping function consistently above the diagonal dashed line implies that clock time lags behind development, and therefore development is late in clock terms. If height is measured relative to biological age, the curves in Fig. 2 are aligned as shown in Fig. 5. Differences in height acceleration after registration correspond to pure amplitude variation in growth patterns. The mean of the registered curves in Fig. 5 is a much better summary than the mean of the unregistered data in Fig. 1.

3. *The Nondurable Goods Index and More Derivatives*

Valuable information may be gained by plotting one derivative of a functional observation against another, a technique called phase/plane plotting. The US index of nondurable goods production is an important economic indicator, and considerable insight may be gained by plotting the acceleration of this index against its rate of increase. If the index had a smooth trend plus a sinusoidal seasonal variation, this plot would be approximately circular with suitable choice of scales. The typical year 1970 in Fig. 6 has three main cycles, of which the spring cycle is rather smaller than the summer and fall/winter cycles. Comparisons of plots for different years is instructive. During the stable period 1964–6, there is little year-to-year change, but the period 1973–5 in Fig. 6 shows considerable instability, due primarily to the end of the Vietnam War. A typical plot for a more recent year, 1994, shows that the whole pattern has changed, indicating structural changes in the methods and economics of production. For further details, see Ramsay and Ramsey (2001).

4. *Functional Principal Components Analysis*

Many FDA methods are adaptations of classical multivariate methods such as principal components analysis (PCA), linear modeling, and analysis of variance (see *Linear Hypothesis* and *Multivariate Analysis: Overview*). Functional PCA demonstrates the way in which a set of functional data varies from its mean, and, in terms of these modes of variability, quantifies the discrepancy from the mean of each individual functional datum. Figure 7 (fig. 6.2 in Ramsay and Silverman (1997)) shows the first four principal component weight curves for temperature records for 35 Canadian weather stations. The first mode of variability is overall temperature, but with larger effects in the winter; the second corresponds to variation in annual range of temperature; the third partly to a time-shift of the overall cycle, and the fourth to short, hot summers and long, mild winters, or vice versa. Individual weather stations can then be

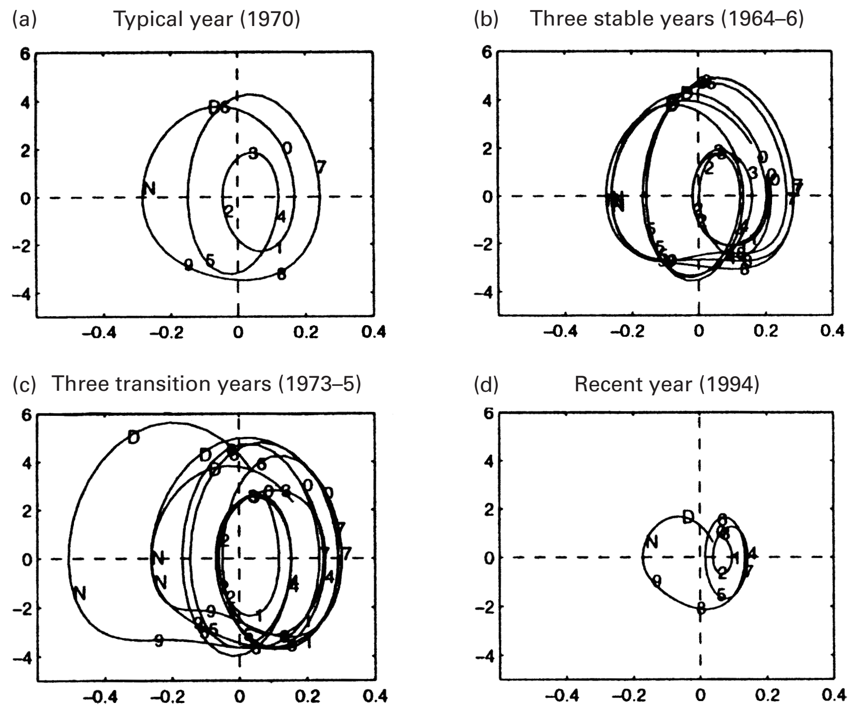

Figure 6
A phase/plane plot of *US index of nondurable goods production*. Each panel plots second derivative on vertical axis against first derivative on horizontal axis. Plots show structure of seasonal variation in index, and how structure varies with time

characterized according to their scores on these four criteria. The interpretability of the components can often be improved by a suitable rotation of the original principal components; see Ramsay and Silverman (1997, Chap. 6).

5. *A Functional Linear Model and Regularization*

An example of functional linear modeling is provided by the study of the dependence of the acceleration of the lower lip (lipacc) in speech on neural activity, as measured by electromyographical (EMG) recording. It is reasonable to suppose that the neural activity at time s continues to affect lip movement for only a short period of time, which we denote by δ and this leads to a model of the form

$$lipacc(t) = \alpha(t) + \int_{t-\delta}^{t} EMG(s)\beta(s,t)\, dt + \varepsilon(t),$$

developed by Malfait et al. (2000). The bivariate function $\beta(s, t)$ plays the same role as the regression coefficient in standard linear regression, and quantifies the relationship between the independent functional variable EMG, and the dependent functional variable lip. One particular issue is the appropriate choice of the time lag δ the length of time within the model that neural activity can continue to affect movement. Models for increasing values of δ correspond to nested linear models, and in this case a study of the amount of variation explained demonstrates that there is little effect beyond a lag of about half a second.

An important theoretical and practical issue is the necessity or otherwise of incorporating some form of smoothing or regularization into a functional analog of a classical multivariate procedure. In principal components analysis, the principal components, such as those displayed in Fig. 7, are smoothed or regularized and in some procedures, such as canonical correlation analysis (see *Multivariate Analysis: Overview*) or certain types of linear modeling for the lip/EMG data, regularization is necessary to give any meaningful results at all; see Leurgans et al. (1993), and Ramsay and Silverman (1997, Chaps. 11, 12).

6. *Modeling with Derivatives: A Central Theme*

It is the smoothness of the processes generating functional data that differentiates this type of data

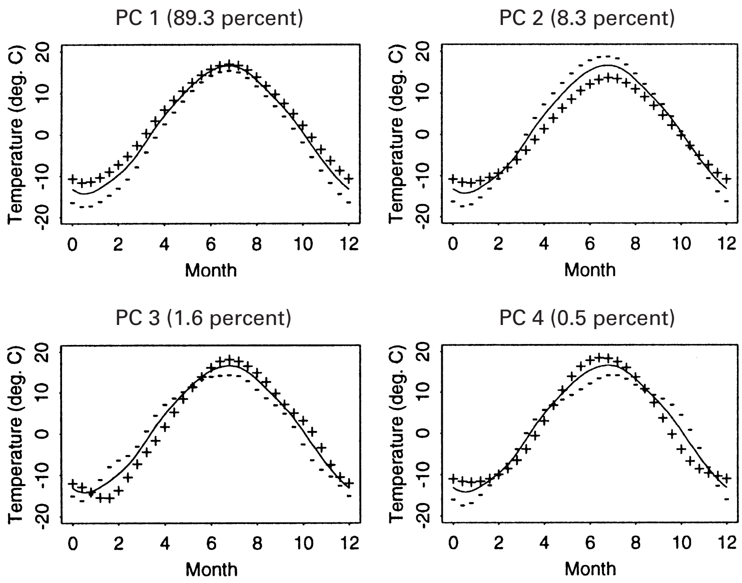

Figure 7
A principal components analysis of mean temperature curves for 35 Canadian weather stations
Notes: Solid line is the mean curve taken across all stations. The effects of adding (+) and subtracting (−) a suitable multiple of each component are also shown.

from more classical multivariate observations. This smoothness means that we can work with the information in the derivatives of functions or images. We looked at acceleration in the growth data, found something interesting in the joint variation of the velocity and acceleration of nondurable goods production, and used derivatives to define roughness penalties; see Ramsay and Silverman (1997, Chap. 4) for the weather data.

We may also use derivatives to construct models for data. Such models are called differential equations, and are part of the standard model-building toolkit in

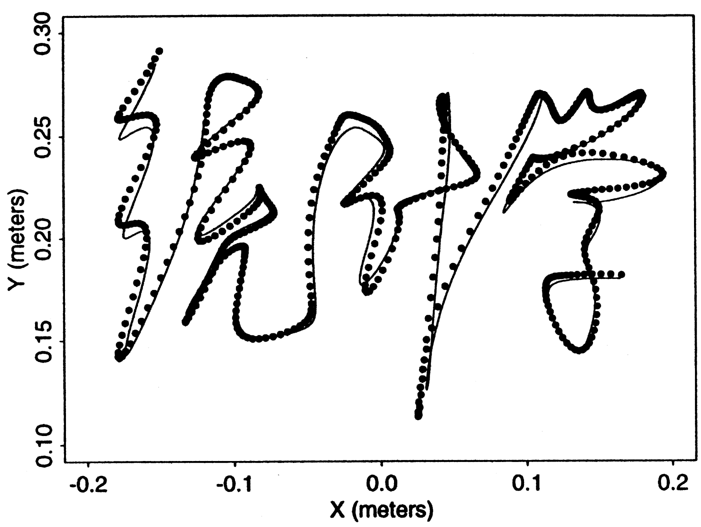

Figure 8
Observations of the tip of the pen while writing the word ‘statistics’ in Mandarin
Notes: Dots indicate 2400 observations; solid line is an approximation based on a second-order linear differential equation.

the natural sciences and engineering. Differential equation models are often called dynamic models, since they are especially effective at representing processes unfolding in time. Ramsay (2000) developed a linear differential equation for handwriting involving velocity, acceleration, and the third derivative of the tip of the pen. Figure 8 shows a sequence of pen positions during the writing of the word 'statistics' in Mandarin, and also shows as a solid line its approximation on the basis of the dynamic equation estimated from 50 such samples of Chinese script. The equation not only captured the essential features of this particular sample, but was also able to model a fair amount of the sample-to-sample variation as well. And, finally, it also provided an excellent account of the velocity, acceleration, and jerk functions at the same time. Ramsay and Silverman (1997, Chaps. 13, 14) take up this topic of dynamic modeling with functional data, including an account of a technique called 'principal differential analysis,' developed by Ramsay (1995).

7. *Relationships with Other Branches of Statistics*

Of course, statisticians have been exploring curves for a very long time, and there are many older statistical methodologies that can continue to inform and enrich an FDA. First of all, FDA usually begins with some kind of nonparametric regression (e.g., see *Exploratory Data Analysis: Multivariate Approaches (Nonparametric Regression)*) in order to replace discrete and possibly noisy curve values by smooth, continuous functions. Density estimation also involves data smoothing. Typically, time series analysis (Brillinger 1981, Box and Jenkins 1994) considers long sequences of curve values corresponding to equally spaced time values that are more or less stationary; that is, the data show little overall trend, and the covariance among neighboring values does not change much from one time region to another. However, methods for non-stationary processes are also under active development, and there are strong structural links between times series methods such as state–space modeling and FDA methods such as smoothing. Longitudinal data analysis or multilevel modeling (Diggle et al. 1994, Searle et al. 1992) usually looks at shorter sequences of curve values sampled over many cases, and studies the structure of between- and within-curve variation. It seems likely that this field and FDA will merge in a number of useful ways. Multivariate statistics has many methods developed specifically for functional data that have preceded FDA, but which have not explicitly exploited the smoothness of functional data or made other uses of derivatives (see *Probability Density Estimation*; *Longitudinal Data*; *Multivariate Analysis: Overview*).

8. *Conclusion*

By the smoothness of a functional observation we have meant the possibility that one or more derivatives, such as velocity or acceleration, may be examined or used in the modeling process. Otherwise, FDA makes only minimal assumptions about the structure of variation within and between curves. We have seen, however, that teasing apart amplitude and phase variation is a central issue, and one that has largely been ignored until recently. Certain types of models considered to be fairly routine and well understood in multivariate statistics, such as linear regression, have many more variants in the functional domain, and are being researched actively. Software for FDA is also being developed, and it is likely that FDA will soon be considered as a routine item in a data analyst's black bag of instruments and nostrums. Further information on FDA and related issues may be obtained in Pinheiro and Bates (2000), Ramsay and Silverman (1997), and Simonoff (1996).

Bibliography

Box G E P, Jenkins G M 1994 *Times Series Analysis: Forecasting and Control*, 3rd edn. Prentice-Hall, Englewood Cliffs, NJ

Brillinger D R 1981 *Time Series: Data Analysis and Theory*. Holden Day, San Francisco

Diggle P J, Liang K-Y, Zeger S L 1994 *Analysis of Longitudinal Data*. Clarendon Press, Oxford, UK

Falkner F (ed.) 1960 *Child Development: An International Method of Study*. Karger, Basle, Switzerland

Leurgans S E, Moyeed R A, Silverman B W 1993 Canonical correlation analysis when the data are curves. *Journal of the Royal Statistical Society, Series B* **55**: 725–40

Malfait N, Ramsay J O, Froda S 2000 The historical functional linear model. Unpublished manuscript. McGill University, Montreal, PQ

Pinheiro J C, Bates D M 2000 *Mixed-effects Models in S and S-PLUS*. Springer, New York

Ramsay J O 1995 Principal differential analysis: Data reduction by differential operators. *Journal of the Royal Statistical Society* **58**: 495–508

Ramsay J O 1998 Estimating smooth monotone functions. *Journal of the Royal Statistical Society, Series B* **60**: 365–75

Ramsay J O 2000 Functional components of variation in handwriting. *Journal of the American Statistical Association* **95**: 9–15

Ramsay J O, Dalzell C 1991 Some tools for functional data analysis (with discussion). *Journal of the Royal Statistical Society* **53**: 539–72

Ramsay J O, Li X 1998 Curve registration. *Journal of the Royal Statistical Society, Series B* **60**: 351–63

Ramsay J O, Ramsey J B 2001 Functional data analysis of the dynamics of the monthly index of non-durable goods production. *Journal of Econometrics* (in press)

Ramsay J O, Silverman B W 1997 *Functional Data Analysis*. Springer, New York

Searle S R, Casella G, McCulloch C E 1992 *Variance Components*. Wiley, New York
Simonoff J S 1996 *Smoothing Methods in Statistics*. Springer, New York

J. O. Ramsay and B. W. Silverman

Functional Equations in Behavioral and Social Sciences

Theories in the behavioral, social, and natural sciences are often formalized by equations involving unknown functions, i.e., by functional equations. For instance, a theorist may be reluctant to make specific assumptions regarding the form of functions involved in a mathematical model, but the qualitative formulation of the model itself may impose constraints on the initially unknown functions. Such constraints often reduce the possibilities and occasionally are so severe that they restrict the possible forms to a few. A recent general reference to functional equations is Aczél and Dhombres (1989) where references to earlier and still useful surveys are given.

In the simplest cases there is one unknown, real-valued, function φ. An example, named after the famous French mathematician A. L. Cauchy, is the Cauchy equation $\varphi(x+y) = \varphi(x)+\varphi(y)$, where x, y are in the set of real numbers $\mathbb{R}$. More restrictive domains are sometimes studied. With no further restrictions, the solutions to the Cauchy equation can be wild. But assuming that φ is monotonic, or continuous, or bounded over a finite interval, the solution reduces to $\varphi(x) = cx$ for some constant c. By taking $\psi = \exp\varphi$, we get $\psi(x+y) = \psi(x)\psi(y)$, which is the so-called 'lack of memory property' and it is easy to see from the previous case that the strictly monotonic solutions are $\psi(x) = e^{cx}$, $c \neq 0$.

A celebrated psychophysical example is the connection between empirical just noticeable differences and G. T. Fechner's hypothesis that sensation corresponds to subjective differences being equal. By one interpretation of what he meant, one is led to the family of functional equations, named Abel equations after the famous Norwegian mathematician N. H. Abel, $\eta[x+g(k,x)]-\eta(x) = h(k)$, where the variable x and the parameter k are in the set of nonnegative real numbers $\mathbb{R}_+$. For the special case of Weber's law where $g(k, x) = (k-1)x$, this equation reduces to $\eta(kx) - \eta(x) = h(k) = \eta(k)-\eta\ (1)$, i.e., with $\varphi(z) = \eta(\exp z) - \eta\ (1)$, $k' = \log k$, $x' = \log x$, we have $\varphi(k'+x') = \varphi(k')+\varphi(x')$, which if k' is treated as a variable is Cauchy's equation. The restriction of strict monotonicity yields $\eta(x) = a\ln x+b$, $a \neq 0$, b constants, as the solution. For references and a general discussion of closely related issue see Falmagne (1985, Chap. 4).

Sometimes a functional equation has multiple, qualitatively different solutions. Yet the scientist arriving at the functional equation has the strong intuition that only one of these solutions is really appropriate for the scientific problem in question. Whenever this happens, the challenge is to discover additional behavioral properties that seem to be empirically correct and that serve to isolate the desired solution. A reason for fully determining the several solutions is that the same functional equation may arise in an entirely different empirical context and, for that context, one of the previously unacceptable solutions may be appropriate. So the complete characterization is clearly of interest.

As will be seen, functional equations arise in the social sciences in at least three main ways. One occurs when one knows how to measure numerically the same attribute in two different ways, which is the case more often than not. Then the two measures are related by an unknown strictly increasing function. An empirical law linking the two underlying measurement structures manifests itself as an equation restricting that unknown function.

Functional equations also arise when some invariance condition holds. The Fechner hypothesis above is an example. Others are given later.

Economic aggregation problems are a third source of functional equations. Consistent aggregation both rules out some *ad hoc* aggregation functions and leads to families of functions that are indeed consistent. Despite the intuitive appeal of additive aggregation, that assumption is inconsistent with the most common production functions used in economics. Examples are given of production functions and of permissible aggregation rules.

1. Independent Measures of an Attribute

1.1 A Physical Example: Mass Measures

Consider mass measurement where one decides which of two objects x and y is more massive by observing which arm of an equal-arm beam balance drops (in a vacuum). This provides a mass ordering. One can construct a numerical measure m of mass, unique up to multiplication by a positive constant (ratio scale), by placing pairs of objects, denoted by $x \circ y$, on the pans. Under reasonable, empirically-testable assumptions m is additive over the operation of combining, i.e., $m(x \circ y) = m(x)+m(y)$. One can also construct a measure unique up to power transformations (log-interval scale) by varying the velocity of objects and determining which has greater momentum. Under empirically reasonable assumptions, the measure of momentum is multiplicative in powers of a mass and a velocity measure, i.e., $\alpha m'(x)^{\beta}v'(x)^{\beta}$, where $\alpha, \beta > 0$ but are otherwise unspecified. It can be shown that the two

International Encyclopedia of the Social & Behavioral Sciences ISBN: 0-08-043076-7