Wavelets: the key to intermittent information?

By B. W. Silverman and J. C. Vassilicos

In recent years, there has been an explosion of interest in wavelets, in a wide range of fields in science and engineering and beyond. Our aim in organizing the Royal Society Discussion Meeting on which this volume is based was to bring together researchers in wavelets from disparate fields, both in order to provide a showcase for a wider audience, and to encourage cross-fertilization of ideas. The meeting, held on 24 and 25 February 1999, attracted a large and enthusiastic audience. Apart from the main papers collected here, there was lively discussion as well as some very interesting contributed posters, many of which will be published elsewhere in the scientific literature.

Of course, many of the ideas behind wavelets are by no means new. What has been achieved in the last ten to fifteen years is the common mathematical framework into which these have been incorporated; the understanding of the connection between discrete filter banks and wavelet bases for functions in continuous time; the development of software for wavelets and many related methods; and a broadening perspective showing just how widely wavelets can be used.

The basic ideas are simply stated. In broad terms, a wavelet decomposition provides a way of analysing a signal both in time and in frequency. If f is a function defined on the whole real line, then, for a suitably chosen *mother wavelet* function ψ, we can expand f as

$$f(t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} w_{jk} 2^{j/2} \psi(2^j t - k)$$

where the functions $\psi(2^j t - k)$ are all orthogonal to one another. The coefficient w_{jk} conveys information about the behaviour of the function f concentrating on effects of scale around 2^{-j} near time $k \times 2^{-j}$. This wavelet decomposition of a function is closely related to a similar decomposition (the discrete wavelet transform or DWT) of a signal observed at discrete points in time.

The DWT can be calculated extremely quickly, and has the property of being very good at *compressing* a wide range of signals actually observed in practice—a very large proportion of the coefficients of the transform can be set to zero without appreciable loss of information, even for signals that contain occasional abrupt changes of level or other behaviour. It is this ability to deal with heterogeneous and intermittent behaviour that makes wavelets so attractive. Classical methods of signal processing depend on an underlying notion of stationarity, for which methods such as Fourier analysis are very well adapted. If one moves away from stationary behaviour, particularly towards the intermittent behaviour with sharp individual events found in many or most real systems, then methods like wavelets are likely to come into their own. Non-stationarity and intermittency, or sparsity of information, can appear in many different guises, but one can also find in the

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wavelet tool-kit a wide variety of mother wavelets and wavelet-like transforms, each one suited to a specific class of problems. Examples discussed in some length in this issue are ridgelets, harmonic wavelets, complex wavelets and the Mexican hat wavelet.

It has already become clear that in many cases the standard wavelet methodology has to be extended and modified in order to be useful. The paper by Daubechies *et al.* demonstrates how wavelet ideas can be applied to data obtained on irregular spatial grids, by making use of the ideas of the lifting scheme. They retain the basic principle in wavelet analysis of splitting the data into 'detail' and 'signal', but the concept of 'level' in the transformation is defined by reference to the local geometry of the surface. An extension in a different direction is explored by Candes and Donoho, who deal with certain multi-dimensional surfaces by using *ridgelets*, which are only localized in one direction in space. They investigate the properties of ridgelets, and in particular demonstrate that they may have much better compression performance than standard multi-dimensional wavelets on images actually observed in practice, especially those with discontinuities along straight edges.

Kingsbury provides a description of a particular implementation of complex wavelets to image processing. He shows how complex wavelet filter banks overcome some of the limitations of classical wavelet filter banks and are both shift invariant and directional. Nicolleau and Vassilicos use the Mexican hat wavelet to define a new fractal dimension, the eddy capacity, and they discuss in what sense this eddy capacity is a direct geometrical measure of scale-invariant intermittency. Of course, wavelets are not a panacea, and Prandoni and Vetterli investigate ways that wavelets are not the best approach to the compression of piecewise polynomial signals.

Why are wavelets useful in statistical applications? This question was approached both from a general and a specific point of view. Johnstone explains and explores the way in which sparsity of representation allows for improvement in the statistical performance of estimation methods. Combined with other insights into the way that wavelets and related families represent phenomena of genuine interest, this provides a unified approach to understanding the merits of wavelet approaches. Silverman gives some directions in which wavelet methodology, especially in statistics, can be extended beyond the standard assumptions, and among other things considers ways in which wavelets can be used to estimate curves and functions observed with correlated noise and on irregular grids. The time-frequency properties of wavelet analysis make it appropriate for wavelets to be used in the analysis of time series, and Nason and von Sachs explore this area, drawing examples from anaesthesiology and other areas.

Several authors brought to the meeting their expertise in specific areas of application. Ramsey gives a wide ranging discussion of the relevance of wavelets to economics. One of his examples concerns the notion that transactions of different sizes have effects over different time scales; the capacity of wavelets to analyse processes in both scale and time is important here. Pen's paper gives a practical illustration of the way that a wavelet approach gives a good denoising of images obtained in astronomy. Newland demonstrates that harmonic wavelets are a very efficient tool for the time-frequency analysis of transient and intermittent data in vibrations and acoustics. Arneodo *et al.* use the continuous wavelet transform to educe an underlying multiplicative structure from velocity data of homogeneous isotropic turbulence of high Reynolds number. This structure is found to be intermittent and dependent on Reynolds number, and may be the kinematic reflection of a cascading process in the turbulence. Moving from the physical to the biological sciences, Field explains that wavelet-like structures may be involved in the mammalian visual system because of their efficiency in capturing the sparse statistical structure of natural scenes.

The genuine interdisciplinary nature of wavelet research is demonstrated by the breadth of contributions to the meeting and to this volume. It is perhaps too early to answer definitely the question posed in the title of the meeting, whether wavelets are indeed *the* key to understanding and modelling heterogeneous and intermittent phenomena, but it is already clear that they have a very substantial role to play, and that the Discussion Meeting has played an important part in developing this. We are extremely grateful to our fellow organizers, Ingrid Daubechies and Julian Hunt, for their suggestions in constructing the programme, to Irene Moroz and Guy Nason for organizing the poster sessions of contributed papers, to the Novartis Foundation for their support, and to the Royal Society for sponsoring and hosting the meeting.