Time Series
Exercises and Problems, Sheet 2

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1. A time series with periodic component can be constructed from

$$Y_t = U_1 sin(2\pi\nu t) + U_2 cos(2\pi\nu t),$$

where U_1 and U_2 are independent random variables with zero means and variances σ^2. The constant ν determines the period or the time that it takes the process to make one complete cycle. Show that this series is weakly stationary with acf

$$\rho_h = cos(2\pi\nu h).$$

2. Consider a process consisting of a linear trend with an additive noise term consisting of independent random variables ϵ_t with zero means and variances σ^2, that is,

$$Y_t = \beta_0 + \beta_1 t + \epsilon_t,$$

where β_0 and β_1 are fixed constants. Prove that Y_t is nonstationary, but that the first difference series $\nabla Y_t = Y_t - Y_{t-1}$ is second-order stationary.

3. A vector time series $Y_{\mathbf{t}} = (Y_{t1}, \ldots, Y_{tp})^T$ contains as its components p univariate time series. For the stationary case, the $p \times 1$ mean vector is

$$\mu = E(Y_{\mathbf{t}}) = (\mu_{t1}, \ldots, \mu_{tp})^T$$

and the autocovariance function can be defined as a function of the multidimensional lag vector $\mathbf{h} = (h_1, \ldots, h_p)^T$ as

$$\gamma_{\mathbf{h}} = E(Y_{\mathbf{s+h}} - \mu)(Y_{\mathbf{s}} - \mu).$$

For a two-dimensional process, for example, we have

$$\gamma_{(h_1,h_2)} = E(Y_{s_1+h_1, s_2+h_2} - \mu)(Y_{s_1,s_2} - \mu).$$

A concept used in geostatistics is that of the *variogram*, defined for a spatial process $Y_{\mathbf{s}}, \mathbf{s} = (s_1, s_2), s_1, s_2 = 0, \pm 1, \pm 2, \ldots$ as

$$V_Y(\mathbf{h}) = \frac{1}{2} E\left((Y_{\mathbf{s}+\mathbf{h}} - Y_{\mathbf{s}})^2\right).$$

Show that, for a stationary process, the variogram and the autocovariance function can be related through

$$V_Y(\mathbf{h}) = \gamma_{\mathbf{0}} - \gamma_{\mathbf{h}},$$

where $\mathbf{0} = (0, 0)$.

4. Determine which of the following processes are causal and/or invertible. Here, ϵ denotes white noise.
(a) $Y_t + .2Y_{t-1} - .48Y_{t-2} = \epsilon_t$
(b) $Y_t + 1.9Y_{t-1} + .88Y_{t-2} = \epsilon_t + .2\epsilon_{t-1} + .7\epsilon_{t-2}$
(c) $Y_t + .6Y_{t-2} = \epsilon_t + 1.2\epsilon_{t-1}$
(d) $Y_t + 1.8Y_{t-1} + .81Y_{t-2} = \epsilon_t$
(e) $Y_t + 1.6Y_{t-1} = \epsilon_t - .4\epsilon_{t-1} + .04\epsilon_{t-2}$.

5. For an ARMA(1,1) process with $|\theta| < 1$, verify that

$$\rho_h = \frac{(1+\theta\phi)(\theta+\phi)}{1+2\theta\phi+\theta^2}\phi^{h-1}, \quad h \geq 1.$$