1. Denote the c.d.f. of a statistic $T = t(\mathbf{X})$ under the simple null hypothesis H_0 by F_{T,H_0}, and assume that F_{T,H_0} is continuous. Put

$$P(\mathbf{x}) = P(T \geq t(\mathbf{x})|H_0).$$

Let $\mathbf{X}$ be a random sample from the distribution specified by the null hypothesis. Show that then the random variable $P(\mathbf{X})$ is uniformly distributed on $[0,1]$.

2. Suppose you have a random sample of size n from an exponential distribution with mean μ. Find the best size-α test of $H_0 : \mu = \mu_0$ against the alternative $H_1 : \mu = \mu_1$, where $\mu_1 > \mu_0$.

3. Suppose you have a random sample $\mathbf{X} = X_1, \ldots, X_n$ of size n from a distribution which is a member of a continuous 1-parameter regular exponential family, so that

$$f(x;\theta) = \exp\left\{\phi(\theta)h(x) + c(\theta) + d(x)\right\}, \quad x \in \mathcal{X},$$

and assume that ϕ is an increasing function. Let $T = t(\mathbf{X}) = \sum_{i=1}^n h(X_i)$. Consider $H_0 : \theta = \theta_0$ and for fixed $\alpha > 0$ choose k_α such that $P_{\theta_0}(t(\mathbf{X}) \geq k_\alpha) = \alpha$. The following test is proposed: Reject H_0 if $t(\mathbf{x}) \geq k_\alpha$.

(a) Let $\theta_1 > \theta_0$. Show that the test is most powerful when testing H_0 against $H_1 : \theta = \theta_1$.

(b) Show that the test is uniformly most powerful when testing H_0 against $H_1 : \theta > \theta_0$.

4. Consider the linear model

$$Y = \beta x + \gamma z + \epsilon,$$

where $\epsilon \sim \mathcal{N}(0, \sigma^2)$, and σ^2 is unknown. Let $\hat{\sigma^2}$ be the maximum-likelihood estimator for σ^2. Let $\theta = (\beta, \gamma)^T$, and denote the estimated variance-covariance matrix of θ by $\hat{V}(\hat{\theta})$; say,

$$\hat{V}(\hat{\theta}) = \hat{\sigma^2} \begin{pmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{pmatrix}.$$

a) Show that for testing $H_0 : \beta\gamma = 1$ the Wald statistic is

$$\frac{(\hat{\beta}\hat{\gamma} - 1)^2}{\hat{\sigma^2}(\hat{\gamma}^2 w_{11} + 2\hat{\beta}\hat{\gamma}w_{12} + \hat{\beta}^2 w_{22})}.$$

b) Derive the Wald statistic for testing $H_0 : \beta = \gamma^{-1}$.

5. Suppose $Y_i \sim N(\beta_0 + \beta_1 {x_i}^\psi, \sigma^2)$ for $i = 1, \ldots, n$, where β_0, β_1, ψ and σ are unknown parameters and where the constants x_i are known. Derive the score test of $H_0 : \psi = 1$ against the alternative $H_1^+ : \psi > 1$.

6. Suppose $X_i \sim N(\mu_x, \sigma_x^2)$ for $i = 1, \ldots, n$, and $Y_j \sim N(\mu_y, \sigma_y^2)$ for $j = 1, \ldots, m$, where μ_x, μ_y, σ_x and σ_y are all unknown. Let S_x^2 and S_y^2 denote the sample variances. Use the pivot $(S_x^2/\sigma_x^2)/(S_y^2/\sigma_y^2)$ to obtain an exact upper $1-\alpha$ confidence limit for $\psi = \sigma_y^2/\sigma_x^2$. How would you construct the corresponding confidence limit for ψ if both μ_x and μ_y were known?

7. Let $X_{(n)}$ be the largest value in a sample of size n drawn from the uniform distribution on $[0, \theta]$. Show that $X_{(n)}/\theta$ is a pivot. Using this pivot, find a $100(1-\alpha)\%$ confidence interval for θ. Discuss how you would test the hypothesis that θ takes a specific value θ_0 for such a sample.