1. Which of the following densities are within an exponential family? Explain your reasoning.

(a)
$$f(x,\theta) = (1-\theta)\theta^x; \quad x = 0, 1, 2, \ldots$$
where $0 < \theta < 1$;

(b)
$$f(x;\alpha,\lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1}, \quad x > 0,$$
where $\alpha > 0, \lambda > 0$;

(c)
$$f(x,\theta) = e^{-(x-\theta)}, \quad x \geq \theta.$$

2. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the Pareto distribution $f(x,\lambda) = \frac{\lambda \alpha^\lambda}{x^{\lambda+1}}$ with $x > \alpha$, $\lambda > 0$, and $\alpha > 0$ known. Find the likelihood function for λ, and find a minimal sufficient statistic for λ.

3. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the log-normal distribution with density
$$f(x,\mu,\phi) = \frac{1}{x\sqrt{2\pi\phi}} \exp\left\{-\frac{1}{2\phi}(\log x - \mu)^2\right\}$$
with $\phi > 0$ (so that $\log X_j \sim \mathcal{N}(\mu,\phi)$). Find a minimal sufficient statistic for the parameter $\theta = (\mu,\phi)$.

4. Suppose $X_1, \ldots, X_n$ are independent and exponentially distributed, each with density function
$$f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}, \qquad x \geq 0.$$
Let $\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i$ and put
$$T = \frac{\overline{X}}{\sqrt{\frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X})^2}}.$$
Show that T is an ancillary statistic. What does this say about t-tests on exponential data?

5. Let $X_1, \ldots, X_n$ be i.i.d. uniform $\mathcal{U}\left[\theta - \frac{1}{2}, \theta + \frac{1}{2}\right]$ random variables.

a) Show that $\left(X_{(1)}, X_{(n)}\right)$ is minimal sufficient for θ.
b) Show that $(S, A) = \left(\frac{1}{2}(X_{(1)} + X_{(n)}), X_{(n)} - X_{(1)}\right)$ is minimal sufficient for θ, and that the distribution of A is independent of θ (so A is an ancillary statistic).
c) Show that any value contained in the interval $\left[x_{(n)} - \frac{1}{2}, x_{(1)} + \frac{1}{2}\right]$ is a maximum-likelihood-estimator for θ.

6. The random variables $X_1, \ldots, X_n$ are independent with geometric distribution $\mathsf{P}(X_i = x) = p(1-p)^{x-1}$ for $x = 1, 2, \ldots$. Let $\theta = p^{-1}$.

(i) Find the maximum likelihood estimator for p. Considering $n = 1$, is it unbiased?

(ii) Show that $\hat{\theta} = \overline{X}$ is the maximum likelihood estimator for θ. Is it unbiased?

(iii) Compute the expected Fisher information for θ.

(iv) Does $\hat{\theta}$ attain the Cramer-Rao lower bound?

7. Let $X_1, \ldots, X_n$ be i.i.d. $U[0, \theta]$, having density

$$f(x; \theta) = \frac{1}{\theta}, \qquad 0 \leq x \leq \theta$$

with $\theta > 0$.

(i) Estimate θ using both the method of moments and maximum likelihood.

(ii) Calculate the means and variances of the two estimators.

(iii) Which one should be preferred and why?

8. Suppose $X_1, \ldots, X_n$ are a random sample with mean μ and finite variance σ^2. Use the delta method to show that, in distribution,

$$\sqrt{n}(\overline{X}_n^{\,2} - \mu^2) \to \mathcal{N}(0, 4\mu^2\sigma^2).$$

What would you suggest if $\mu = 0$?