Stochastic Simulation
Exercises and Problems, Sheet 3

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1. Evaluate the integral

$$\int_0^1 \frac{e^x - 1}{e - 1} dx$$

- by direct calculation
- using the crude Monte Carlo estimator, and determine its variance
- using importance sampling, with importance function $g(x) = x$, and determine the variance of the importance sampling estimator
- using a control variate that is $\mathcal{U}([0,1])$, and determine the variance of the corresponding estimator

2. Show that if X and Y have the same distribution then $Var[(X + Y)/2] \leq Var(X)$, and conclude that the use of antithetic variables can never increase variance (although it need not be as efficient as generating an independent set of numbers).

3. In certain situations a random variable X, whose mean and variance are known, is simulated so as to obtain an estimate of $P(X \leq a)$ for a given constant a. The raw simulation estimator from a single run is

$$I = \mathbf{1}(X \leq a).$$

Because I and X are clearly negatively correlated, a natural attempt to reduce the variance is to use X as a control - and so use an estimator of the form $I + c(X - E[X])$.

- Determine the variance reduction over the raw estimator I that is possible (by using the best c) if $X \sim \mathcal{U}([0,1])$.

- Repeat the above if $X \sim exp(1)$.
- Explain why we knew that I and X were negatively correlated.

4. Suppose we want to estimate

$$\theta = \int_0^1 \sqrt{1 - x^2} dx.$$

- Let $V_1, V_2 \sim \mathcal{U}[-1, 1]$ be independent, and put

  $$I = \mathbf{1}(V_1^2 + V_2^2 \leq 1).$$

  Show that $\mathbf{E}I = \frac{\pi}{4}$. Calculate the variance of I.

- Show that, for $-1 \leq v \leq 1$,

  $$\mathbf{E}(I|V_1 = v) = (1 - v^2)^{\frac{1}{2}},$$

  and calculate the variance of $\mathbf{E}(I|V_1)$.