Problem Sheet 6

1. Given a posterior distribution $\pi(\theta \mid \mathbf{x})$ of a parameter θ given data $\mathbf{x}$, find the Bayes estimate of θ under quadratic loss.

Two independent Poisson processes with rates θ_1, θ_2 are each observed for unit time. The numbers of events observed are x_1 and x_2 respectively. Find the likelihood $L(\psi, \phi; x_1, x_2)$ where $\psi = \theta_1/(\theta_1 + \theta_2)$ and $\phi = \theta_1 + \theta_2$. If the prior distributions of ψ and ϕ are independent with $\psi \sim U(0, 1)$, show that the posterior distribution of ψ is

$$\pi(\psi \mid x_1, x_2) = \frac{(x_1 + x_2 + 1)!}{x_1!\, x_2!} \psi^{x_1}(1 - \psi)^{x_2}, \qquad 0 \leqslant \psi \leqslant 1,$$

and determine the Bayes estimate of ψ under quadratic loss.

2. A population consists of m classes $\theta_1, \ldots, \theta_m$ and it is required to classify an individual on the basis of an observation X. If the observation X is from class θ_i, then it has density function $f(x \mid \theta_i)$, $i = 1, \ldots, m$. The prior probabilities of the classes are $\pi_1, , \ldots, \pi_m$ and the loss in classifying an individual from class i into class j is l_{ij}.

Find the posterior probability $h_i(x)$ of class θ_i and the posterior risk of assigning the individual to class i.

Now consider the case of 0–1 loss, i.e.

$$l_{ij} = \begin{cases} 0 & i = j, \\ 1 & i \neq j. \end{cases}$$

Show that the risk is equal to the probability of misclassification.

Suppose further that $m = 3$, that $\pi_i = 1/3$ for all i, and that X is normally distributed with mean $\theta_i = i$ and variance 1 in class i. Find the Bayes rule for classifying an observation. Use it to classify the observation $x = 2.2$.

3. Consider the linear model

$$Y_i = \beta x_i + \varepsilon_i, \qquad i = 1, \ldots, n,$$

where $\varepsilon_1, \ldots, \varepsilon_n$ are independent $N(0, \sigma^2)$ random variables with σ^2 unknown.

Find the maximum likelihood estimator $\widehat{\beta}$ of β. What is the distribution of $\widehat{\beta}$?

How would you estimate the variance of $\widehat{\beta}$?

Suppose $Y' = \widehat{\beta} x' + \varepsilon'$, denoting the value of Y at $x = x'$, is about to be measured, where $\varepsilon' \sim N(0, \sigma^2)$ independently of $\varepsilon_1, \ldots, \varepsilon_n$. What is the variance of Y'? Construct a 95% confidence interval for Y'.

4. A West Coast publishing company keeps accurate records of its monthly expenditure of advertising and its total monthly sales. For the first 10 months of 1995, the records showed the following.

Advertising (in thousand) (x)	43	44	36	38	47	40	41	54	37	46
Sales (in millions) (y)	74	76	60	68	79	70	71	94	65	78

Note that $\sum x_i = 426, \sum y_i = 735, \sum x_i^2 = 18,416$, and $\sum x_i y_i = 31,763$.

Find the least-squares equation appropriate for the data.

Find 95% confidence intervals for the regression coefficients.

If the company spends \$50,000 for advertising next month, what is its predicted sales? Assume that all other factors can be neglected. Give a 95% prediction interval.

If the company spends nothing for advertising next month, what is its predicted sales? Give a 95% prediction interval. Comment on the result.