1. When X, Y, Z are i.i.d. $\mathcal{N}(0, \theta)$, the distance U from the origin to the point (X, Y, Z) has the *Maxwell distribution*

$$f_U(u; \theta) \propto \frac{u^2}{\theta^{\frac{3}{2}}} e^{-\frac{u^2}{2\theta}}; \quad u > 0.$$

   A statistic $T = T(X_1, \ldots, X_n)$ is called *asymptotically efficient* if its asymptotic variance is equal to the Cramer-Rao lower bound. Show that the statistic $T = \frac{1}{3n} \sum_{i=1}^n U_i^2$ is asymptotically efficient in estimating θ. (**Hint:** Show that f is in the exponential family.)

2. Describe how the *simple likelihood ratio* can be used to construct a uniformly most powerful test of a null hypothesis H_0 against a composite hypothesis H_1, when such a test exists. Illustrate your explanation by constructing a test of the null hypothesis $H_0 : \mu = \mu_0$ against the alternative $H_1 : \mu < \mu_0$ based on a random sample of data from a Poisson distribution with unknown mean μ. What would the corresponding results be if the alternative hypothesis were $H_1 : \mu > \mu_0$?

   Now suppose that the alternative hypothesis is $H_1 : \mu \neq \mu_0$. Give an asymptotic result about the *generalised likelihood ratio* which could be used in order to test the hypothesis $H_0 : \mu = \mu_0$ and calculate the test statistic in this case. If the null hypothesis is $\mu = 5$, the number in the sample is 50 and the sum of the observations is 319, use the asymptotic theory to test the null hypothesis against the general alternative $\mu \neq 5$.

3. Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with mean $\frac{1}{\lambda}$. Prove that

$$P\left(b / \sum_{i=1}^n X_i < \lambda\right) = 1 - \alpha,$$

   where $1 - \alpha = \frac{1}{\Gamma(n)} \int_b^\infty y^{n-1} e^{-y}\, dy$. Let $x_1, x_2, \ldots, x_n$ be observed valued in a sample and let $c = \sum_{i=1}^n x_i$. Deduce that $(b/c_n, \infty)$ is a $100(1-\alpha)\%$ confidence interval for λ.

   For the same data it is required to test $\lambda = \lambda_0$ against the alternative hypothesis $\lambda > \lambda_0$. Show that the p-value for the likelihood ratio test satisfies $p = \frac{1}{\Gamma(n)} \int_0^{\lambda_0 c_n} y^{n-1} e^{-y}\, dy$. Deduce that if $p > \alpha$ then $\lambda_0 \in (b/c_n, \infty)$.

4. Let Z denote a discrete random variable with values in $\{z_2, \ldots, z_n\}$ having one of the two distributions f_1 or f_2. Explain how to construct a likelihood-ratio test for testing the null hypothesis $H_0 : f_1$ *is the true distribution* against the alternative hypothesis $H_1 : f_2$ *is the true distribution*.

   Now suppose that f_1 and f_2 are as follows.

	z_1	z_2	z_3	z_4	z_5
f_0	.2	.3	.1	.3	.1
f_1	.3	.1	.3	.2	.1

Carry out the above test at level $\alpha = .3$. What is the power of this test?

5. An experiment conducted to check the variability in explosion times of detonators intended to detonate simultaneously resulted in the following times to detonation in microseconds:

   2.689, 2.677, 2.675, 2.691, 2.698, 2.694, 2.702,
   2.698, 2.706, 2.692, 2.691, 2.681, 2.700, 2.698.

   Assume that the data come from a normally distributed random sample.

   (a) The specification is that the standard deviation $\sigma \leq 0.007$; is it met? Use a test of significance at level $\alpha = 0.05$.

   (b) Is there any evidence against the hypothesis that the mean explosion time is 2.7 microseconds? Carry out a test of significance at level $\alpha = 0.05$, assuming that the true standard deviation is $\sigma = 0.007$.

   (c) Now test the same hypothesis but do not assume that you know the true underlying variance.

6. (a) Amounts of water filtered with a particular filtration evice using two methods of operation were recorded and these summary statistics calculated: Method 1 had 20 observation, with a sample average of 202.0 and a sample standard deviation of 74.35, whereas for Method 2 there are 29 observations, with a sample average of 278.3 and a sample standard deviation of 79.03. Describe the appropriate likelihood ratio test, and test the hypothesis that the variances are the same under the two methods at level $\alpha = 0.05$.

   (b) A survey of the use of a particular product was conducted in four areas, with a random sample of 200 potential users interviewed in each area. The results were that in the four areas, respectively, x_1, x_2, x_3, and x_4 of the 200 interviewees said that they used the product. Carry out a likelihood ratio test to test whether the proportion of the population using the product is the same in each area, with $\alpha = 0.05$, when $x_1 = 76, x_2 = 53, x_3 = 59$ and $x_4 = 48$, using the large sample approximation for the distribution of the test statistic.