Algorithmic Foundations of Learning

Lecture 2
Maximal Inequalities and Rademacher complexity

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Offline statistical learning: prediction

Given a batch of observations (images & labels) interested in predicting the label of a new image.
Offline statistical learning: prediction

1. Observe training data $Z_1, \ldots, Z_n$ i.i.d. from unknown distribution
2. Choose action $A \in \mathcal{A} \subseteq \mathcal{B}$
3. Suffer an expected/population loss/risk $r(A)$, where

$$a \in \mathcal{B} \rightarrow r(a) := \mathbb{E} \ell(a, Z)$$

with $\ell$ is an prediction loss function and $Z$ is a new test data point

**Goal:** Minimize the estimation error defined by the following decomposition

$$r(A) - \inf_{a \in \mathcal{B}} r(a) = r(A) - \inf_{a \in \mathcal{A}} r(a) + \inf_{a \in \mathcal{A}} r(a) - \inf_{a \in \mathcal{B}} r(a)$$

- excess risk
- estimation error
- approximation error

as a function of $n$ and notions of “complexity” of the set $\mathcal{A}$ of the function $\ell$

**Note:** Estimation/Approximation trade-off, a.k.a. complexity/bias
ERM and Uniform Learning

▶ A natural framework is given by the empirical risk minimization (ERM)

$$a \in \mathcal{B} \rightarrow R(a) := \frac{1}{n} \sum_{i=1}^{n} \ell(a, Z_i)$$

▶ A natural algorithm is given by the minimizer of the ERM

$$A^* \in \arg\min_{a \in \mathcal{A}} R(a)$$

▶ Uniform Learning: The estimation error is bounded by

$$r(A^*) - r(a^*) \leq \sup_{a \in \mathcal{A}} \{ r(a) - R(a) \} + \sup_{a \in \mathcal{A}} \{ R(a) - r(a) \}$$

▶ Statistical Learning deals with bounding the Statistics term (Vapnik 1995)

▶ Generalization Error: $$r(a) - R(a) \approx \frac{1}{n^{(test)}} \sum_{i=1}^{n^{(test)}} \ell(a, Z_i^{(test)}) - \frac{1}{n} \sum_{i=1}^{n} \ell(a, Z_i)$$
Goal: derive bounds in expectation

Goal:

\[ \mathbb{E} \left( r(A^*) - r(a^*) \right) \lesssim \frac{f(\text{dimension})}{n^\alpha} \]

estimation error for ERM

By uniform learning, it suffices to bound the suprema of random processes:

\[ \mathbb{E} g(Z_1, \ldots, Z_n) \leq \frac{f(\text{dimension, complexity of } \mathcal{A})}{n^\alpha} \]

with \( g(Z_1, \ldots, Z_n) = \sup_{a \in \mathcal{A}} \{ r(a) - R(a) \} = \sup_{a \in \mathcal{A}} \left\{ \mathbb{E} \ell(a, Z) - \frac{1}{n} \sum_{i=1}^{n} \ell(a, Z_i) \right\} \)

We aim to derive a uniform, non-asymptotic Law of Large Numbers

In machine learning, dimension can be \( \gg 10^6 \), e.g., number of pixels

Ideally, \( f(\text{dimension}) \ll \text{dimension} \), e.g., \( f(\text{dimension}) \sim \log(\text{dimension}) \)

Ideally, \( \alpha = 1 \) (fast rate)
Hoeffding’s Lemma (Lemma 2.1)

Let $X$ be a bounded random variable $a \leq X \leq b$. Then, for any $\lambda \in \mathbb{R}$ we have

$$
\mathbb{E} e^{\lambda (X - \mathbb{E}X)} \leq e^{\frac{\lambda^2(b-a)^2}{8}}
$$

Proof

- W.l.o.g., take $\mathbb{E}X = 0$. Let $\psi(\lambda) = \log \mathbb{E} e^{\lambda X}$

  $$
  \psi'(\lambda) = \frac{\mathbb{E}[X e^{\lambda X}]}{\mathbb{E} e^{\lambda X}}, \quad \psi''(\lambda) = \frac{\mathbb{E}[X^2 e^{\lambda X}]}{\mathbb{E} e^{\lambda X}} - \left( \frac{\mathbb{E}[X e^{\lambda X}]}{\mathbb{E} e^{\lambda X}} \right)^2
  $$

- $\psi''(\lambda)$ is the variance of $X$ under the distribution $Q(dx) = \frac{e^{\lambda x}}{\mathbb{E} e^{\lambda X}} P(dx)$

- $\psi''(\lambda) = \text{Var}_Q \left( X - \frac{a+b}{2} \right) \leq \mathbb{E}_Q \left[ \left( X - \frac{a+b}{2} \right)^2 \right] \leq \frac{(b-a)^2}{4}$

- Fundamental Thm of Calculus: $\psi(\lambda) = \int_0^\lambda \int_0^\mu \psi''(\rho) d\rho d\mu \leq \frac{\lambda^2(b-a)^2}{8}$
Maximum of finitely many bounded random variables (Proposition 2.2)

Let $X_1, \ldots, X_n$ be $n$ centered random variables bounded in the interval $[a, b]$.

$$\mathbb{E} \max_{i \in [n]} X_i \leq \frac{b - a}{\sqrt{2}} \sqrt{\log n}$$

Proof

- $X = \max_{i \in [n]} X_i$. Exponentiate. Jensen’s ineq. as $x \to e^{\lambda x}$ ($\lambda > 0$) is convex:

  $$\mathbb{E} X = \frac{1}{\lambda} \log e^{\lambda \mathbb{E} X} \leq \frac{1}{\lambda} \log \mathbb{E} e^{\lambda X}$$

- Bound maximum of non-negative numbers by the sum:

  $$\mathbb{E} e^{\lambda X} = \mathbb{E} e^{\lambda \max_{i \in [n]} X_i} = \mathbb{E} \max_{i \in [n]} e^{\lambda X_i} \leq \mathbb{E} \sum_{i=1}^n e^{\lambda X_i} = \sum_{i=1}^n \mathbb{E} e^{\lambda X_i}$$

- Put everything together and use Hoeffding’s lemma ($\mathbb{E} e^{\lambda X_i} \leq e^{\lambda^2 (b-a)^2 / 8}$):

  $$\mathbb{E} \max_{i \in [n]} X_i \leq \frac{1}{\lambda} \log \sum_{i=1}^n e^{\lambda^2 (b-a)^2 / 8} = \frac{1}{\lambda} \log n + \frac{\lambda (b - a)^2}{8}$$

- Optimizing the bound $\alpha/\lambda + \lambda \beta$ over $\lambda > 0$ yields the maximum is at $\lambda = \sqrt{\alpha/\beta}$ and the optimal value $2\sqrt{\alpha\beta} = (b - a) \sqrt{\log n} / 2$
Bound in expectation for finitely-many actions

Bound in expectation (Proposition 2.3)

If the loss function $\ell$ is bounded by $c$, we have

$$E \max_{a \in \mathcal{A}} \{r(a) - R(a)\} \leq c \frac{\sqrt{2 \log |\mathcal{A}|}}{\sqrt{n}}$$

**Proof:** Same as above, using the independence of the data $Z_1, \ldots, Z_n$ (note that for each $a \in \mathcal{A}$, $r(a) - R(a)$ is a centered random variable as $E R(a) = r(a)$)

- Recall wish: $E \sup_{a \in \mathcal{A}} \{r(a) - R(a)\} \leq f(\text{dimension, complexity of } \mathcal{A}) \frac{n^\alpha}{n}$

- The dimension of the data is superseded by the boundedness assumption

- $\alpha = 1/2$, slow rate

- When $|\mathcal{A}| < \infty$, $|\mathcal{A}|$ is a valid notion of complexity of the problem

- When $|\mathcal{A}| = \infty$, upper bound is trivial and we need another notion of complexity
Rademacher complexity

Rademacher complexity (Definition 2.5)

The Rademacher complexity of a set $\mathcal{T} \subseteq \mathbb{R}^n$ is defined as

$$\text{Rad}(\mathcal{T}) := \mathbb{E} \sup_{t \in \mathcal{T}} \frac{1}{n} \sum_{i=1}^{n} \Omega_i t_i$$

where $\Omega_1, \ldots, \Omega_n \in \{-1, 1\}$ are i.i.d. uniform random variables (Rademacher)

- Measures of complexity: describes how well elements in $\mathcal{T}$ can replicate the sign pattern of a uniform random signal $\in \mathbb{R}^n$ (see Problem 1.4)

- Useful properties:
  
  - $\text{Rad}(c\mathcal{T} + v) = |c| \text{Rad}(\mathcal{T})$ (Proposition 2.6)
  
  - $\text{Rad}(\mathcal{T} + \mathcal{T}') = \text{Rad}(\mathcal{T}) + \text{Rad}(\mathcal{T}')$ (Proposition 2.7)
  
  - $\text{Rad}(\text{conv}(\mathcal{T})) = \text{Rad}(\mathcal{T})$ (Proposition 2.8)

  with $\text{conv}(\mathcal{T}) = \left\{ \sum_{j=1}^{m} w_j t_j : w \in \Delta_m, t_1, \ldots, t_m \in \mathcal{T}, m \in \mathbb{N} \right\}$
Rademacher complexity

Massart’s Lemma (Lemma 2.9)

Let $\mathcal{T} \subseteq \mathbb{R}^n$ and $\bar{t} := \frac{1}{|\mathcal{T}|} \sum_{t \in \mathcal{T}} t$. We have

$$\text{Rad}(\mathcal{T}) \leq \max_{t \in \mathcal{T}} \|t - \bar{t}\|_2 \frac{\sqrt{2 \log |\mathcal{T}|}}{n}$$

**Proof:** Similar to ones given above. **Problem 1.5**

Contraction property - Talagrand’s Lemma (Lemma 2.10)

Let $\mathcal{T} \subseteq \mathbb{R}^n$. For each $i \in \{1, \ldots, n\}$, let $f_i : \mathbb{R} \to \mathbb{R}$ be a $\gamma$-Lipschitz function. Then,

$$\text{Rad}((f_1, \ldots, f_n) \circ \mathcal{T}) \leq \gamma \text{Rad}(\mathcal{T})$$

with $(f_1, \ldots, f_n) \circ \mathcal{T} := \{(f_1(t_1), \ldots, f_n(t_n)) \in \mathbb{R}^n : t \in \mathcal{T}\}$

**Proof:** **Problem 1.6** (optional)
Rademacher complexity

Bound in expectation via Rademacher complexity (Proposition 2.11)

\[ \mathbb{E} \sup_{a \in \mathcal{A}} \{ r(a) - R(a) \} \leq 2 \mathbb{E} \text{Rad}(\mathcal{L} \circ \{ Z_1, \ldots, Z_n \}) \]

with

\[ \mathcal{L} := \{ z \in \mathcal{Z} \to \ell(a, z) \in \mathbb{R} : a \in \mathcal{A} \} \]

and

\[ \mathcal{L} \circ \{ Z_1, \ldots, Z_n \} := \{ (\ell(a, Z_1), \ldots, \ell(a, Z_n)) \in \mathbb{R}^n : a \in \mathcal{A} \} \]

Recall wish:

\[ \mathbb{E} \sup_{a \in \mathcal{A}} \{ r(a) - R(a) \} \leq f(\text{dimension}, \text{complexity of } \mathcal{A}) \frac{1}{n^\alpha} \]

Next lecture we will bound the Rademacher complexity in different settings
Proof: Symmetrization

Proof

▶ Let \( \{\tilde{Z}_1, \ldots, \tilde{Z}_n\} \) be a new sample of independent random variables:

\[
r(a) = E \ell(a, Z) = \frac{1}{n} \sum_{i=1}^{n} E \ell(a, \tilde{Z}_i) = \frac{1}{n} \sum_{i=1}^{n} E[\ell(a, \tilde{Z}_i)|S]
\]

▶ By properties of conditional expectations (tower property and others) we get

\[
E \sup_{a \in A} \{r(a) - R(a)\} = E \sup_{a \in A} \frac{1}{n} \sum_{i=1}^{n} \left( E[\ell(a, \tilde{Z}_i)|S] - \ell(a, Z_i) \right)
\]

\[
= E \sup_{a \in A} \frac{1}{n} \sum_{i=1}^{n} E[\ell(a, \tilde{Z}_i) - \ell(a, Z_i)|S]
\]

\[
\leq EE \left[ \sup_{a \in A} \frac{1}{n} \sum_{i=1}^{n} \{\ell(a, \tilde{Z}_i) - \ell(a, Z_i)\} \right] |S|
\]

\[
= E \sup_{a \in A} \frac{1}{n} \sum_{i=1}^{n} \{\ell(a, \tilde{Z}_i) - \ell(a, Z_i)\}
\]

\[
= E \sup_{a \in A} \frac{1}{n} \sum_{i=1}^{n} \Omega_i \{\ell(a, \tilde{Z}_i) - \ell(a, Z_i)\}
\]

\[
\leq 2 E \sup_{a \in A} \frac{1}{n} \sum_{i=1}^{n} \Omega_i \ell(a, Z_i) = 2 E \text{Rad}(L \circ \{Z_1, \ldots, Z_n\})
\]