Traditionally, **STATISTICS** is taught via *asymptotic* results, for $n \to \infty$:

- **Law of Large Numbers**
- **Central Limit Theorem**, yielding
  - Confidence bounds
  - Hypothesis testing

In this course we have developed **non-asymptotic** results, for $n < \infty$:

- **Uniform Law of Large Numbers**
  - Notions of complexity to bound generaliz. error of ERM _algorithm_
- **Confidence bounds**
  - Analysis of algorithms (upper bounds with high-probability)
  - Design of _algorithms_ (UCB)
- **Hypothesis testing** (Today’s lecture)
  - Lower bounds holding for any _algorithm_

**STATISTICS** lays the foundation of **ALGORITHMS** for machine learning
Hypothesis Testing and Lower Bounds

- **Data:** random variable $X \in \mathcal{X}$
- **Hypotheses:**
  - $X \sim P$ (null hypothesis $H_0$)
  - $X \sim Q$ (alternative hypothesis $H_1$)
- **Test:** any function $f : \mathcal{X} \to \{0, 1\}$
- **Errors:**
  - Type I: if $f(X) = 1$ when $X \sim P$
  - Type II: if $f(X) = 0$ when $X \sim Q$

Any test commits one type of error with strictly positive probability unless $P$ and $Q$ have disjoint support under the reference measure $\rho$.

Neyman Pearson (Lemma 16.1)

For any function $f : \mathcal{X} \to \{0, 1\}$ we have

$$P(f(X) = 1) + Q(f(X) = 0) \geq \int \rho(dx) \min\{p(x), q(x)\}$$

and the equality is achieved by the Likelihood Ratio Test $f^* := 1_{q \geq p}$
Total Variation Distance

**Neyman Pearson Lemma:**

- No matter how we choose the decision rule $f$, we cannot make a decision with probability of error on either $P$ or $Q$ smaller than $\int \rho(dx) \min\{p(x), q(x)\}$
- **Structural limitation** of what we can hope to achieve statistically based on the “amount of information” in the problem
- The greater the overlap between $P$ and $Q$, the more difficult the problem is
- There is a notion of distance behind the scenes...

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**Total variation distance (Definition 16.2)**

\[
\|P - Q\|_{tv} = \sup_{E} |P(E) - Q(E)|
\]

\[
= \frac{1}{2} \int \rho(dx)|p(x) - q(x)|
\]

\[
= 1 - \int \rho(dx) \min\{p(x), q(x)\}
\]

To prove lower bounds on sum of errors, enough to upper bound $\|P - Q\|_{tv}$
Kullback-Leibler Divergence

- In statistics, often data is $X_1, \ldots, X_n$ i.i.d. ($\mathbb{P} = \bigotimes_{i=1}^{n} P_i$ and $\mathbb{Q} = \bigotimes_{i=1}^{n} Q_i$)
- The total variation distance does not factorize under product measures
- The Kullback-Leibler divergence (not a distance!) does factorize instead

### Kullback-Leibler divergence (Definition 16.3)

$$KL(\mathbb{P}, \mathbb{Q}) = \begin{cases} \int \rho(dx)p(x) \log \frac{p(x)}{q(x)} & \text{if } \mathbb{P} \ll \mathbb{Q} \\ +\infty & \text{otherwise} \end{cases}$$

### Properties of Kullback-Leibler divergence (Proposition 16.4)

1. **Gibbs’ inequality:** $KL(\mathbb{P}, \mathbb{Q}) \geq 0$ with equality if and only if $\mathbb{P} = \mathbb{Q}$
2. **Chain rule for product measures:** $$KL(\mathbb{P}, \mathbb{Q}) = \sum_{i=1}^{n} KL(P_i, Q_i)$$
3. **Pinsker’s inequality:** $\|\mathbb{P} - \mathbb{Q}\|_{tv} \leq \sqrt{\frac{1}{2} KL(\mathbb{P}, \mathbb{Q})}$
Corollary 16.5

- **Data:** Let \( X_1, \ldots, X_n \in \mathcal{X} \)

- **Hypotheses:** \( \mathcal{P} \) (null \( H_0 \)) or \( \mathcal{Q} \) (alternative \( H_1 \))

- **Test:** \( f : \mathcal{X}^n \rightarrow \{0, 1\} \)

\[
P(f(X_1, \ldots, X_n) = 1) + Q(f(X_1, \ldots, X_n) = 0) \geq 1 - \sqrt{\frac{1}{2} \text{KL}(\mathcal{P}, \mathcal{Q})}
\]

If \( X_1, \ldots, X_n \) are independent, then

\[
P(f(X_1, \ldots, X_n) = 1) + Q(f(X_1, \ldots, X_n) = 0) \geq 1 - \sqrt{\frac{1}{2} \sum_{i=1}^{n} \text{KL}(P_i, Q_i)}
\]

- **“Amount of information”:** Function of \( n \) and \( \text{KL}(P_i, Q_i), i \in [n] \)
Back to the Multi-Armed Bandit Problem

At every time step \( t = 1, 2, \ldots, n \):

1. Choose an action \( A_t \in \mathcal{A} \)
2. A data point \( Z_t \) is sampled independently from an unknown distribution
   - **Bandit**: \( Z_t \) is not revealed
3. Suffer a loss \( \ell(A_t, Z_t) = -Z_{t,A_t} \)

Vectors \( Z_t \)'s are indep., but observed data \((A_1, Z_1, A_1), \ldots, (A_n, Z_n, A_n)\) are not!

**Proposition 16.7**

- Two bandit models (\( \mu \) and \( \nu \)): rewards for arm \( a \) either \( P_{\mu,a} \) or \( P_{\nu,a} \)
- Fix an algorithm \( A_1, \ldots, A_n \)
- \( P_{\mu} \) and \( P_{\nu} \) probab. each model assigns to \((A_1, Z_1, A_1), \ldots, (A_n, Z_n, A_n)\)

\[
\text{KL}(P_{\mu}, P_{\nu}) = \sum_{a=1}^{k} \text{KL}(P_{\mu,a}, P_{\nu,a}) E_{\mu} N_{n,a}
\]
Theorem 16.6

Let $n \geq k - 1$. For any algorithm there exists a $k$-armed bandit problem with

$$\mathbb{E}R_n \geq c\sqrt{(k - 1)n}$$

where $c$ is a universal constant.

- UCB achieves quasi-optimal distribution-independent pseudo-regret.
- Using similar ideas (but more involved), one can prove that UCB achieves optimal distribution-dependent pseudo-regret.
- Ideas can be generalized to multiple hypothesis testing...

Fano’s Inequality (Theorem 16.9)

Let $\mathbf{P}_1, \ldots, \mathbf{P}_m$ be probability measures such that $\mathbf{P}_\mu \ll \mathbf{P}_\nu$ for any $\mu, \nu \in [m]$

$$\inf_{f} \max_{\mu \in [m]} \mathbf{P}_\mu(f(X) \neq \mu) \geq 1 - \frac{1}{m^2} \sum_{\mu, \nu=1}^{m} \text{KL}(\mathbf{P}_\mu, \mathbf{P}_\nu) + \log 2 \over \log(m - 1)$$
“New science is based on maximum likelihood rather than certainty”

Arthur C. Clarke and Gentry Lee, Rama Series Book 2, 1989