1. We are interested in performing inference about the parameters of an internet traffic model.

(a) The arrival rate $\Lambda$ for packets at an internet switch has a log-normal distribution $\text{LogNormal}(\mu, \sigma)$ with parameters $\mu$ and $\sigma$. The $\text{LogNormal}(\mu, \sigma)$ probability density is

$$p_\Lambda(\lambda; \mu, \sigma) = \frac{1}{\lambda \sqrt{2\pi\sigma^2}} \exp \left( -\frac{(\log(\lambda) - \mu)^2}{2\sigma^2} \right).$$

Show that if $V \sim \mathcal{N}(\mu, \sigma^2)$ and we set $W = \exp(V)$ then $W \sim \text{LogNormal}(\mu, \sigma)$.

(b) Given an arrival rate $\Lambda = \lambda$, the number $N$ of packets which actually arrive has a Poisson distribution, $N \sim \text{Poisson}(\lambda)$. Suppose we observe $N = n$. Show that the likelihood $L(\mu, \sigma; n)$ for $\mu$ and $\sigma$ is

$$L(\mu, \sigma; n) \propto \mathbb{E}(\Lambda^n \exp(-\Lambda)|\mu, \sigma).$$

(c) Give an algorithm simulating $\Lambda \sim \text{LogNormal}(\mu, \sigma)$ using $Y \sim \mathcal{N}(0, 1)$ as a base distribution, and explain how you could use simulated $\Lambda$-values to estimate $L(\mu, \sigma; n)$ by simulating values for $\Lambda$.

(d) Suppose now we have $m$ iid samples

$$\Lambda^{(j)} \sim \text{LogNormal}(\mu, \sigma), j = 1, 2, ..., m$$

for one pair of $(\mu, \sigma)$-values. Give an importance sampling estimator for $L(\mu', \sigma'; n)$ at new parameter values $(\mu', \sigma') \neq (\mu, \sigma)$, in terms of the $\Lambda^{(j)}$’s.

(e) For what range of $\mu'$, $\sigma'$ values can the $\Lambda^{(j)}$-realisation be safely ’recycled’ in this way?

2. Let $X = (X_0, X_1, \ldots)$ be a homogeneous Markov chain taking values in a discrete state space $\Omega$, with transition matrix $P = (p_{ij})_{i,j \in \Omega}$.

(a) Show that if the Markov chain is irreducible, and $p_{ii} > 0$ for some $i \in \Omega$, then the chain is aperiodic.

(b) Consider the homogeneous Markov chain $(X_0, X_1, \ldots)$ with $X_n \in \{1, \ldots, m\}$ and transition matrix

$$p_{ij} = \frac{1}{m} \min \left( 1, \frac{p(j)}{p(i)} \right)$$
for $i \neq j$ and

$$p_{ii} = 1 - \frac{1}{m} \sum_{j \neq i} \min \left( 1, \frac{p(j)}{p(i)} \right)$$

where $p$ is a probability mass function on $\{1, \ldots, m\}$ with $p(i) > 0$ for all $i = 1, 2, \ldots, m$ and $X_0 = 1$.

(i) Show that the Markov chain is irreducible and aperiodic, and admits $p$ as invariant distribution.

(ii) Propose an algorithm to simulate the Markov chain $(X_0, X_1, X_2, \ldots)$ using independent random variables $Y_k \sim U\{1, \ldots, m\}$ and $U_k \sim U[0, 1]$ for $k = 1, 2, \ldots$

3. Here is an algorithm converting a non-negative number $x \in [0, 1)$ to its binary expansion.

Let $b$ be the binary representation of $x$. Compute the first $I$ binary places as follows. Let $i = 1$ and $y = 2x$. If $y$ is greater than or equal one set $b_i = 1$ otherwise set $b_i = 0$; let $x = y - b_i$. If $x$ is now zero or $i = I$ then stop (as either there are no more non-zero places, or we have reached the limit of our number of digits), otherwise increase $i$ by one and repeat.

(a) Write an R function implementing this algorithm. Your function should take as input a single non-negative number $x$ between 0 and 1 and return the corresponding binary representation. Represent the binary number as a vector, so for example decimal 0.125 becomes \(c(0, 0, 1)\) in binary.

(b) At what binary place do Rs numerical values for 0.3 and 0.1+0.1+0.1 differ?

(c) Adapt your function to take two positive integers $0 < p < q$ as input, and return the binary expansion of $p/q$ exactly.

4. Consider a sequence of observations $x_1, \ldots, x_n$. Let $m_i$ and $s_i^2$ denote the mean and sample variance of the first $i$ observations $i \leq n$. How many operations (additions, subtractions, multiplications or divisions) are needed to calculate the sequence of means $m_1, \ldots, m_n$, if each mean is calculated separately?

(a) Derive an expression for $m_{i+1}$ in terms of $m_i$ and $x_{i+1}$ and write an R function that calculates $m_1, \ldots, m_n$ using this sequential formula. How many operations will this function use? \(\text{[Hint: it is important for speed to initialise your output vector with the correct length at the start using numeric(), rather than appending one answer at a time.]}\)

(b) Now consider the sequence of sample variances $s_1^2, \ldots, s_n^2$. Find an expression for $s_{i+1}^2$ in terms of $s_i^2$, $m_i$, $m_{i+1}$ and $x_{i+1}$. Write an R function to evaluate the sample variances using a sequential method.
(c) (Optional.) Write a function to calculate the sample means non-sequentially (using a loop or, for example, \texttt{sapply()}). How long does it take to run when \( n = 10^3, 10^4, 10^5 \)?