

On building and fitting models parameterised by a partial order

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Coauthors

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Thanks R, Rgraphviz, snow, nem, and probably more.

17/11/09

Data (with Dr Nicholas Karn and Dr David Johnson)

12th Century Acts: witness lists

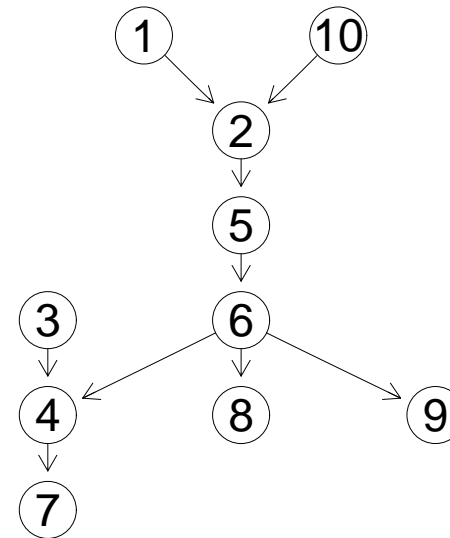
- | | |
|---|-------|
| [1] William, Archbishop of Canterbury | |
| [2] Roger, Bishop of Salisbury | (2) |
| [3] William, Giffard, bishop of Winchester, 1100-1129 | (1) |
| [4] Bernard, Bishop of St David's | (6) |
| [5] William, de Warelwast, bishop of Exeter | (5) |
| [6] Urban, bishop of Llandaff | (8) |
| [7] Geoffrey, Rufus, Bishop of Durham | |
| [8] Robert, de Sigillo, Bishop of London | (10)* |
| (Richard, de Belmeis I, bishop of London, 1108-1127) | |
| [9] Miles, of Gloucester, earl of Hereford 1141-1143 | |
| [10] Henry, de Port | |
| [11] Walter, de Amfreville | |
| [12] William, de Folis | |
| [13] Roger, de Port | |
| [14] William, de Port | |

Covariates: where and when signed (+/-), beneficiaries, ...

Example (heavily and selectively thinned)

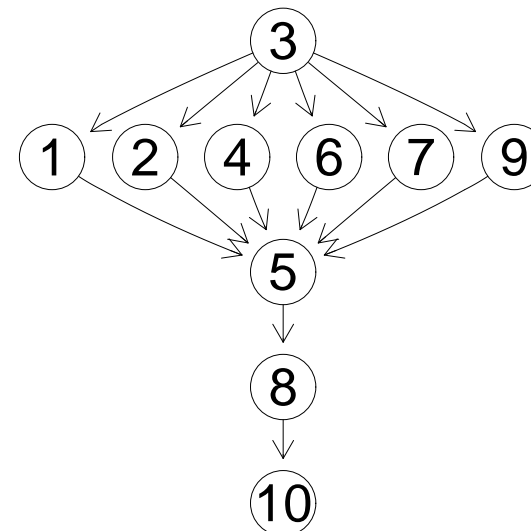
Witness lists (bishops of interest, 1119-1121)

[1119]	5	6	4	7		
[1120]	3	4				
[1121]	1	2				
[1121]	10	1	2	5	6	8
[1121]	1	10	2	5	6	9
[1121]	1	2				
[1121]	10	1	2			



Witness list (bishops of interest, 1127-1129)

[1127]	9	10				
[1127]	2	9	10			
[1127]	2	1	6	5	8	10
[1127]	2	6				
[1127]	2	9	6	10		
[1127]	2	9	6	10		
[1129]	7	10				
[1129]	6	7	4	10		
[1129]	3	4	2			



Partial orders

A PO $H = (V, <)$, elements $x, y \in V$.

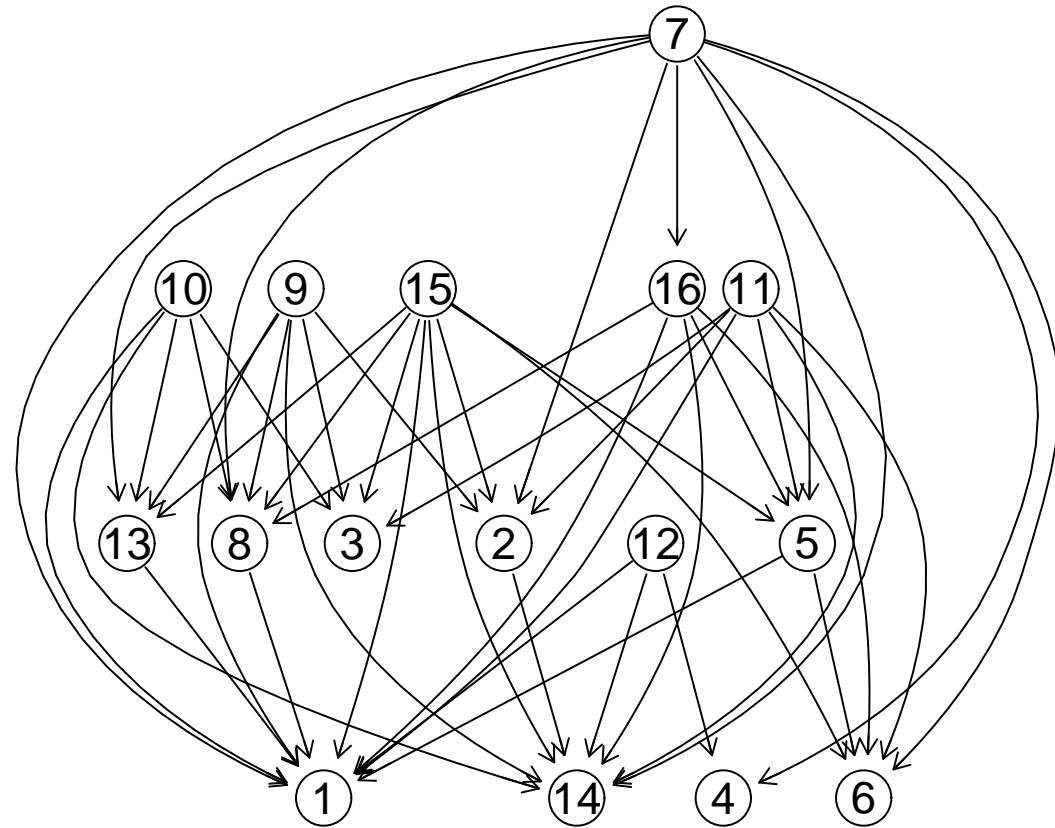
H is a transitive DAG:

irreflexive (no loops)

transitive ($x < y, y < z \Rightarrow x < z$).

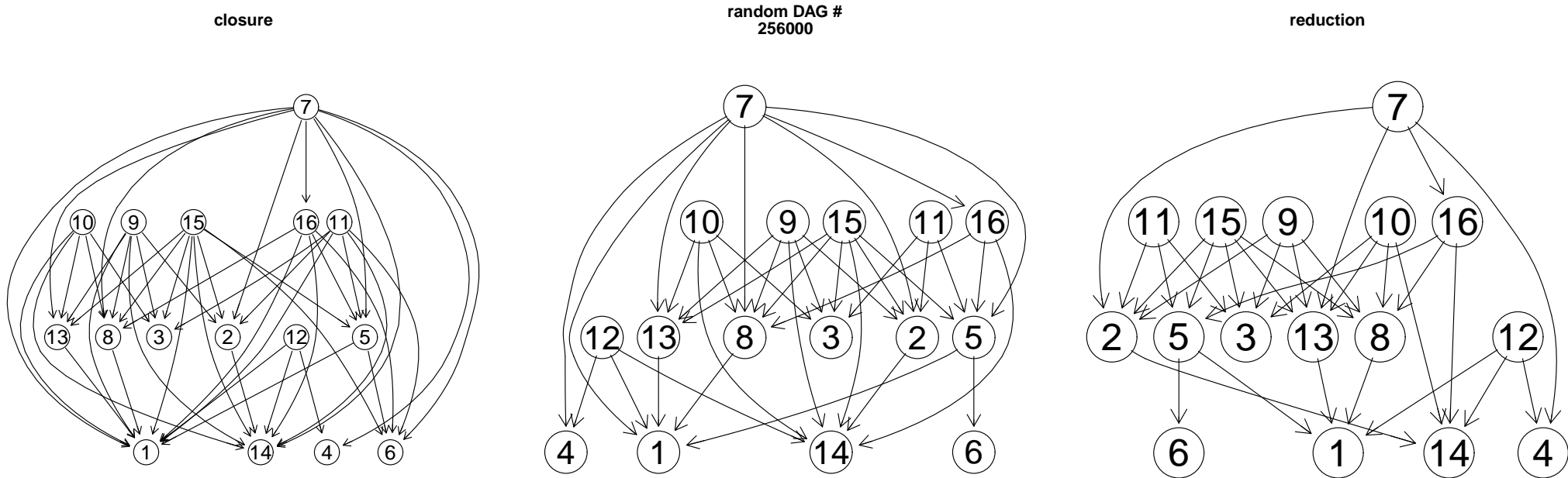
DAG or incidence matrix

$(x, y) \in H, H_{xy} = 1 \Leftrightarrow x < y$ in H .



Suborder, Complete order, Linear extension, Chains (and antichains).

Depth, Width, Dimension.



(left) Transitive closure D^T of DAG D . PO H is closed $H^T = H$.

(right) Transitive Reduction D^t : smallest $d \subseteq D^T$ with $d^T = D^T$.
 D^t is unique (Aho et al. 1972).

$\mathcal{P}_{[n]}^T$ = partial orders on $[n] = \{1, 2, \dots, n\}$.

Note: if $D_+ = D + (x, y)$ with reduction D^t and closure D^T
(La Poutre et al. 1988) compute D_+^t and D_+^T efficiently.

Observation model (static order, ignores mistaken order, list correlation)

Parameter:

Unknown true partial order (status/rank) H

For $o \subseteq [n]$ suborder is $H \downarrow o$.

$\mathcal{L}(H)$ = set of all linear extension of PO H

Data: in act i

n_i witnesses labels $o_i = \{y_{i,1}, y_{i,2}, \dots, y_{i,n_i}\}$, $o_i \subseteq [n]$

Data $y_i = (y_{i,1}, y_{i,2}, \dots, y_{i,n_i})$ respects $H = ([n], <)$

Observation model

$Y_i \sim \text{Uniform}(\mathcal{L}(H \downarrow o_i))$

$L(h; y_i) = \Pr(Y_i = y_i | H = h, O = o_i)$ so $L(h; y_i) = 1/|\mathcal{L}(h \downarrow o_i)|$

Independent lists: $L(h; y) = \prod_i L(h; y_i)$

Berrenwinkel et al 2007 on 'conjunctive Bayesian networks'

MLE is PO nearest complete admitting the observed linear extensions.

Prior distributions for partial orders (I)

See Brightwell (1993) 'Models for Random Partial Orders' for overview.

Properties of Uniform($\mathcal{P}_{[n]}^T$).

almost surely depth 3 (and width about $n/2$), Krietman et al 1975

no parameter controlling typical depth

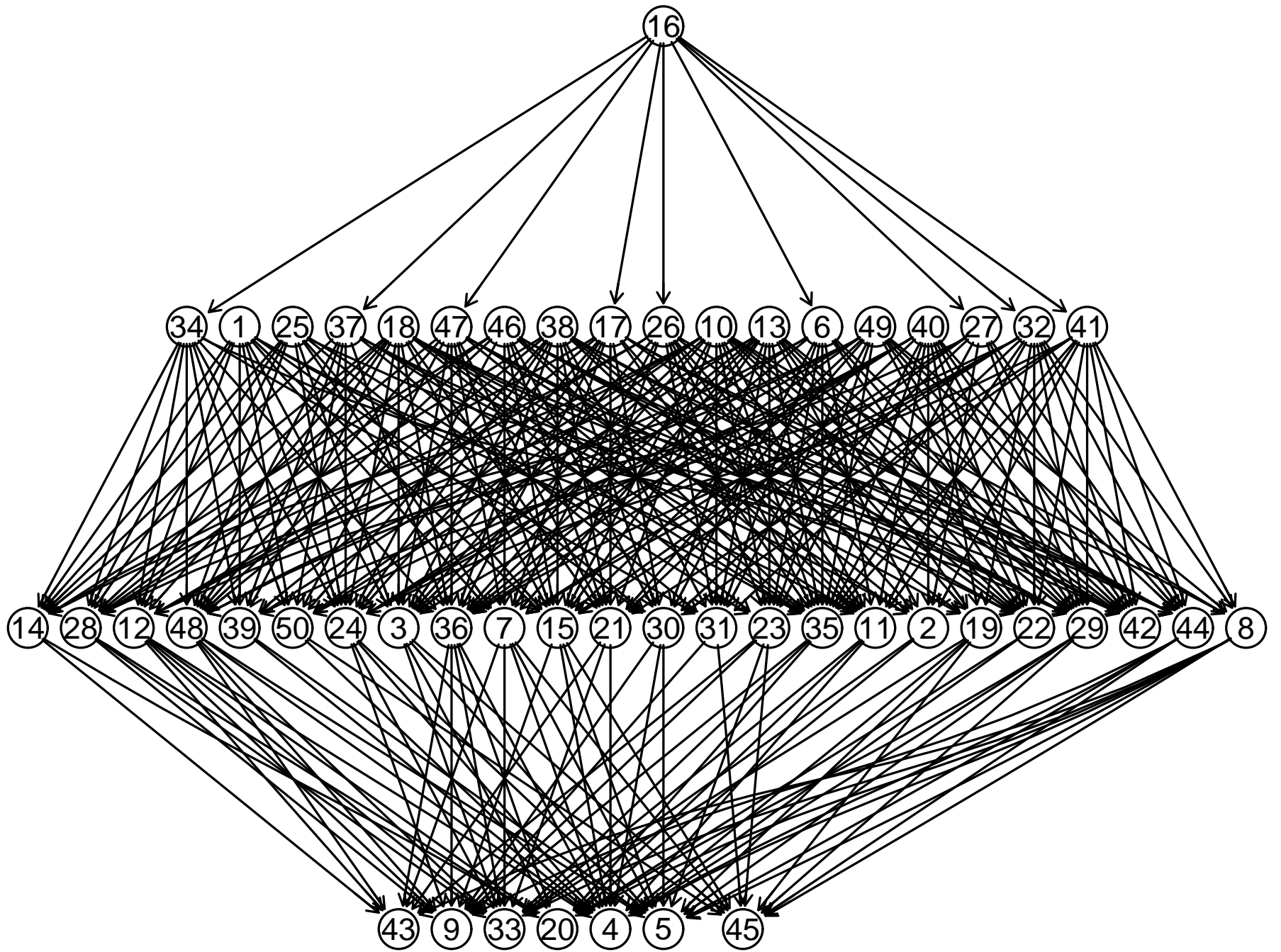
not consistent for marginalisation

Let π_H be a family of probability distributions $\pi_{H,n}(h) = \Pr\{H = h\}$ over $\mathcal{P}_{[n]}^T$ defined for each $n = 1, 2, \dots$. The family π_H is consistent if

$$\pi_{H,n}(h \downarrow [m]) = \pi_{H,m}(h \downarrow [m])$$

for each $n = 1, 2, \dots$, $h \in \mathcal{P}_{[n]}^T$ and all $1 \leq m < n$.

Proof (BWS): there are 19 PO on 3 actors and 3 PO on 2. 19 is not divisible by 3.



Prior distributions for partial orders (II)

Random K -dimensional orders

Actor i location $Z_i \sim \text{Uniform}(0, 1)^K, i = 1, 2, \dots, n$

$$H = I(Z) \quad I(Z)_{ij} = \prod_{k=1}^K \mathbb{I}_{Z_{ik} < Z_{jk}}$$

Dimension at most K .

(Winkler 1985) Random K -dimensional orders are consistent for marginalisation.

Still too punitive on depth (even at $K = 2$).

Example extension: correlate the Z_{ik} 's over k ;

simulate $W_i \sim \text{MVN}(0, \Sigma)$ ($\Sigma_{aa} = 1, \Sigma_{ab} = r$) iid $Z_i = \Phi^{-1}(W)$.

$\text{corr}(Z_{i,k}, Z_{i,k'}) = (6/\pi) \arcsin(r/2)$, parameterize $\text{corr}() \rightarrow r$.

If $K = n$ sample space is all PO's.

Simulation Uniform

\mathcal{D}_n set of all DAGs on the vertex set $V = [n]$.

Let a partial order $H \in \mathcal{P}_n^T$ be given and let

$$S(H) = \{D \mid D^T = H, D \in \mathcal{D}_n\}$$

be the set of DAGs with closure H . Then

$$|S(H)| = 2^{|H| - |H^t|}.$$

Proof: every DAG in $S(H)$ contains H^t as a subgraph and is itself a subgraph of H .

Any subset of arcs in $H \setminus H^t$ can be added to H^t to create a DAG.

MCMC if $d \in \mathcal{D}_n$, $\pi_D(d) \propto 2^{d^t - d^T}$ and $D \sim \pi_D$ then $D^T \sim \text{Uniform}(\mathcal{P}_n^T)$.

Suppose DAG $D^{(m)} = c$.

1. Set $d = c$ then $(x, y) \sim \text{Uniform}([n] \times [n])$ and set $d_{xy} = 1 - c_{xy}$.

2. If d is cyclic $\alpha_D(d|c) = 0$, else

$$\alpha_D(d|c) = \min \left(1, \frac{2^{|d^t| - |d^T|}}{2^{|c^t| - |c^T|}} \right).$$

$D^{(m+1)} = d$ w.p. $\alpha_D(d|c)$ and otherwise $D^{(m+1)} = D^{(m)}$.

Simulation K -dimensional

Un-normalised $\pi_H(h)$ for K -dimensional order is not available

$$\pi_H(h) = \int_{I(z)=h} \pi_Z(z) dz$$

Prior simulation from definition of $H \sim \pi_H(h)$.

Posterior $\pi_{Z|Y}(z|y) \propto \pi_Z(z)L(h; y)$, $z \in [0, 1]^{nK}$, $h = I(z)$

MCMC simulation: suppose $Z^{(m)} = z$ and $h = I(z)$ at step m ;

1. Set $z' = z$ then $(i, k) \sim \text{Unif}[n] \otimes [K]$ and set $z'_{ik} \sim p(z_{ik}|z_{-(ik)})$

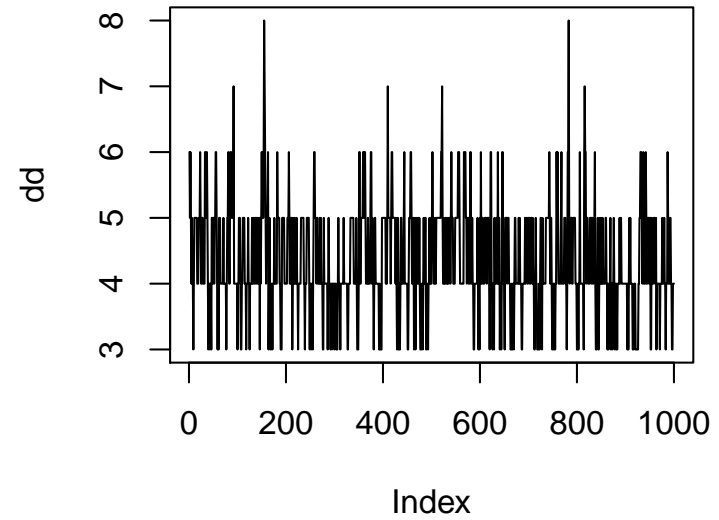
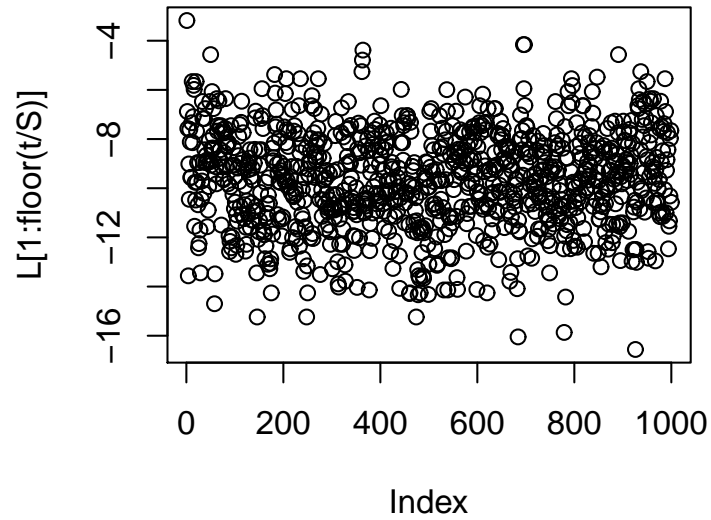
2. $h' = I(z')$ and

$$\alpha_Z(z'|z) = \min(1, L(h'; y)/L(h; y)).$$

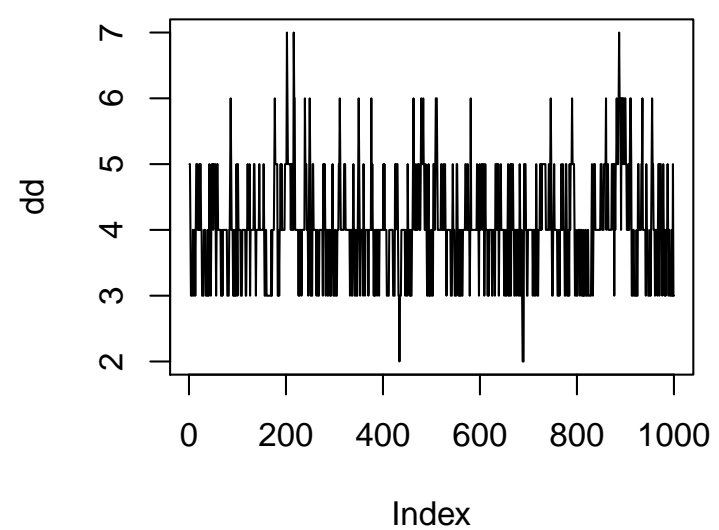
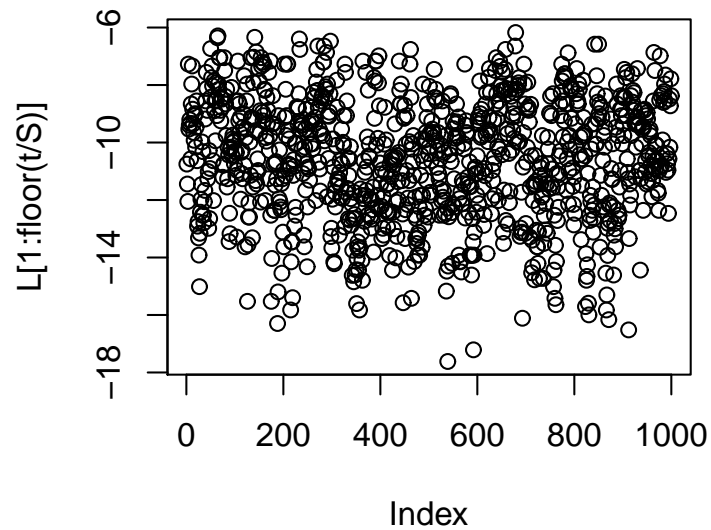
$Z^{(m+1)} = z'$ w.p. $\alpha_Z(z'|z)$ and otherwise $Z^{(m+1)} = Z^{(m)}$.

$H^{(m)} = I(Z^{(m)})$, $m = 0, 1, 2, \dots$ and $H^{(m)} \xrightarrow{D} \pi_{H|Y}(\cdot|y)$.

Results (Uniform prior only so far)



MCMC (left) Loglikelihood (right) depth in (upper) 1119-1121 and (lower) 1127-1129

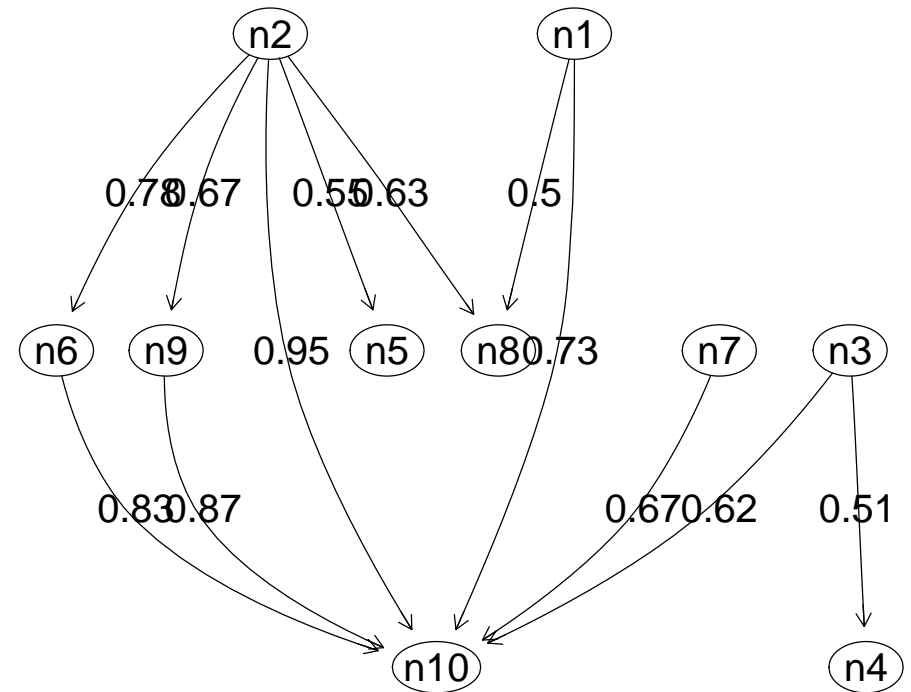
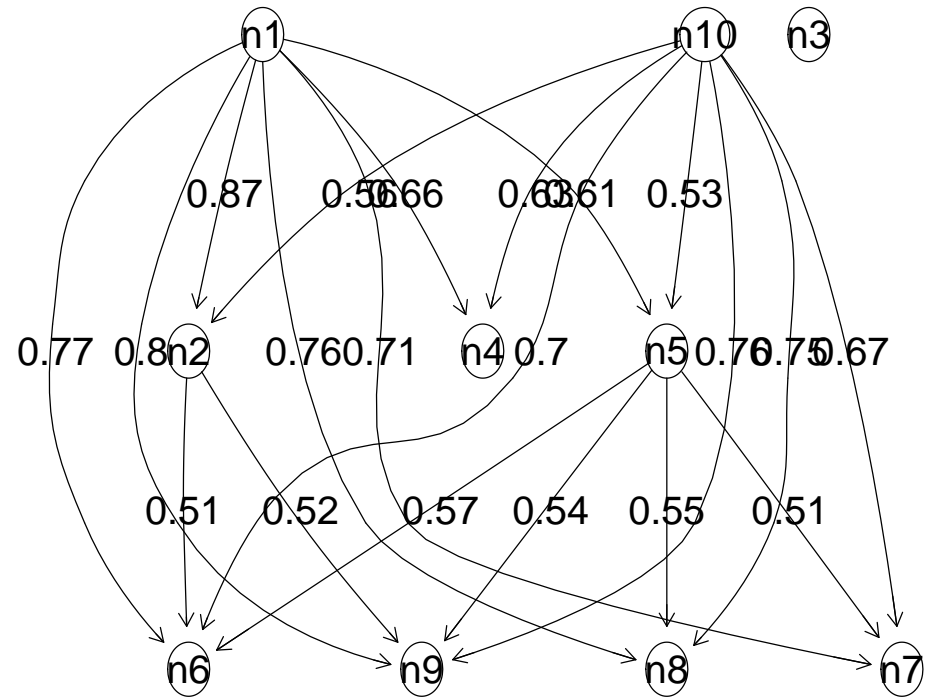


Witness lists (bishops of interest, 1119-1121)

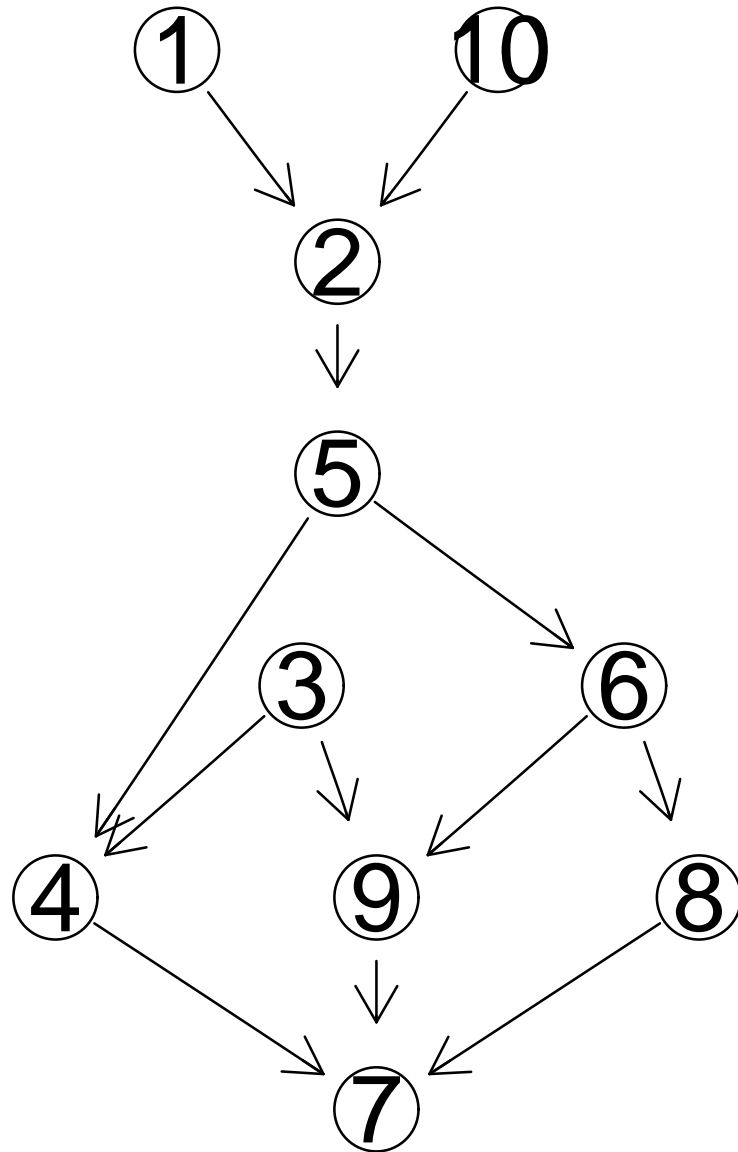
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Witness list (bishops of interest, 1127-1129)

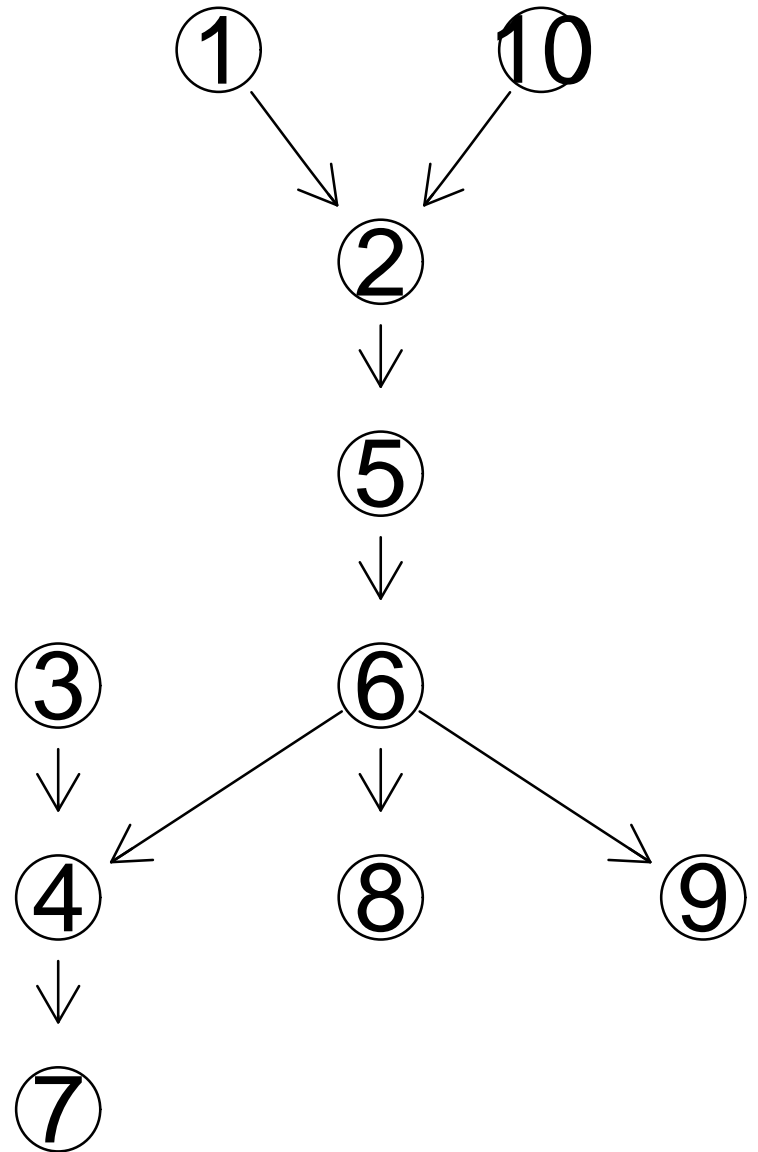
[1127]	9	10			
[1127]	2	9	10		
[1127]	2	1	6	5	8 10
[1127]	2	6			
[1127]	2	9	6	10	
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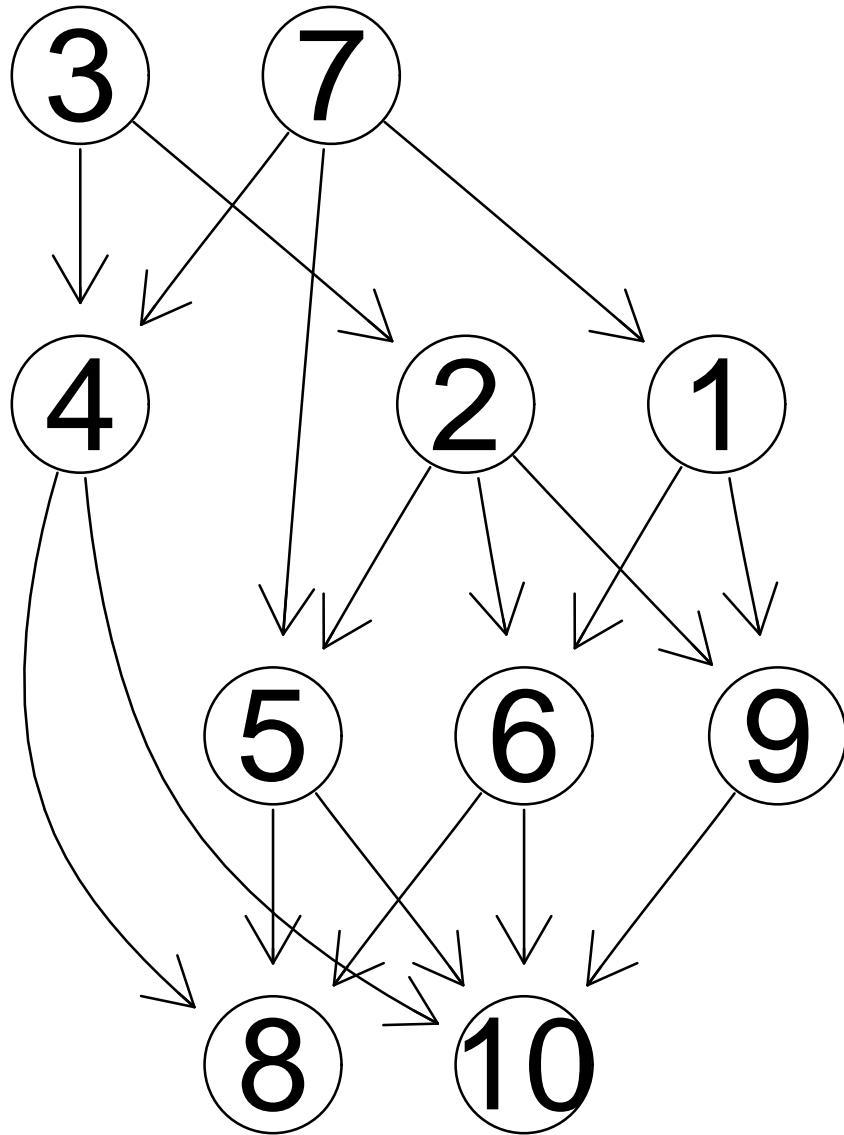
mode



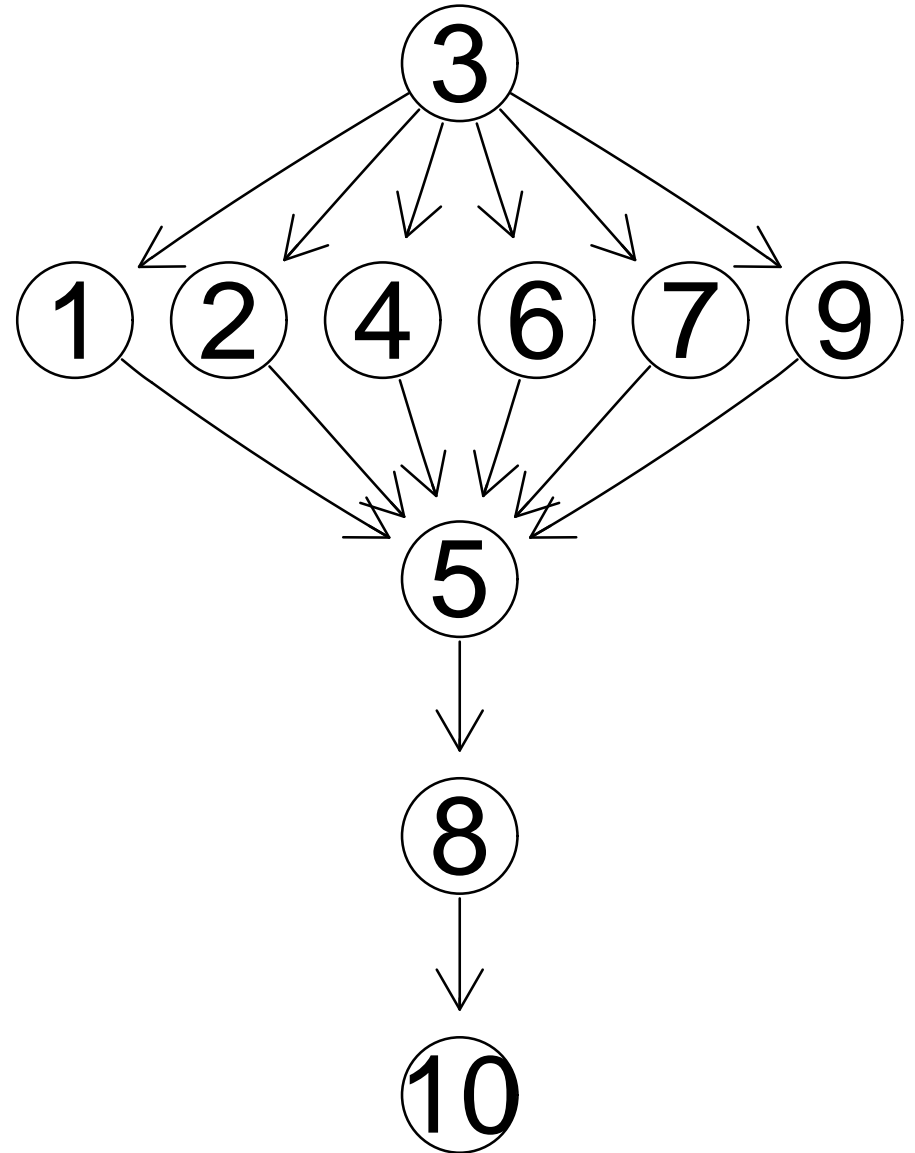
allowed



mode



allowed



Likelihood evaluation and generically parallelising a single serial MCMC chain

$$L(h; y_i) = 1/|\mathcal{L}(H \downarrow o_i)|$$

For $H \in \mathcal{P}_{[n]}^T$, and $V \subset [n]$, let $e_V = |\mathcal{L}(H \downarrow V)|$

Let $U(H)$ be the set of possible 'top' nodes in H .

$$e_{[n]} = \sum_{v \in U(H)} e_{[n] \setminus v}$$

so count LE's recursively. Still time consuming.

Can parallelise accept reject: $D^{(m)} = c$ and k processors.

Let $(d^{(i)}, A^{(i)})$ be candidate (from c) on proc $i = 1, 2, \dots, k$ and and 0-reject/1-accept.

1. Compute $(d^{(1)}, A^{(1)}), (d^{(2)}, A^{(2)}), \dots, (d^{(k)}, A^{(k)})$ in parallel.

2. Let $j = \min\{i : A^{(i)} = 1, i = 1, 2, \dots, k\} \cup \{k + 1\}$.

Set $D^{(m+i)} = c$ for $i = 1, 2, \dots, j - 1$. If $j < k + 1$ set $D^{(m+j)} = d^{(j)}$.

<http://www.stats.ox.ac.uk/~nicholls>

for example (in R) and preprint (geophysical app).

A. Sohn, "Parallel N-ary Speculative Computation of Simulated Annealing" IEEE Trans. Parallel Distrib. Syst., 6, 997-1005 (1995), and earlier.

Easy error takes $H^{(m)} = g, g \in \mathcal{P}_n^T$

1. $(x, y) \sim \text{Uniform}([n] \times [n])$ and if $(x, y) \notin g$, and $g+(x, y)$ acyclic, $h = (g+(x, y))^T$
and if $(x, y) \in g^t$ then $h = (g^t - (x, y))^T$.

2. Compute α etc

Not reversible as $g = \{(1, 3), (2, 3)\}$ and $(x, y) = (1, 2)$ since $h = \{(1, 2), (2, 3), (1, 3)\}$
and $(h^t - (x, y))^T = \{(2, 3)\} \neq g$.

