# Part A Simulation and Statistical programming HT15

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## Lecture 12: Solving Linear Systems.

## Overview for lecture 12

1. R commands for matrices and vectors (reference slides)
2. Solving linear systems $Ax = b$.
   (a) Forwards and Backwards substitution
   (b) Solving $Ax = b$ for full rank $A$ using LU factorization
   (c) Regression.
   (d) Over-determined systems. Numerical stability and QR factorization.

## Solving linear systems

Suppose $A$ is a real $n \times p$ matrix of rank $p$ with $p \leq n$, and entries $a_{i,j}$, and $b$ is an $n \times 1$ real vector.

Many important numerical problems reduce to

solve $Ax = b$ for $x$.

If $p < n$, then the system is over-determined. We come back to this case later. We will look at how the equations $Ax = b$ may be solved when $p = n$ so that $A^{-1}$ exists and $x = A^{-1}b$.

R has a function `solve(A)` returning $A^{-1}$ so we could compute

`x=solve(A)%*%b`.

We will see that this is inefficient and numerically unstable, and find that the best method depends on the properties of $A$.

## Forward and Backward elimination

Suppose $A$ is lower triangular so that $a_{i,j} = 0$ for $i > j$. Solve $Ax = b$ for $x$ using forward substitution. Chop the $n$ equations in $Ax = b$ into blocks

$$A = \begin{pmatrix} a_{11} & 0_{1\times(n-1)} \\ A_{21} & A_{22} \end{pmatrix}$$

Here $A_{21} = A_{2:n,1}$ is $(n-1)\times 1$ and $A_{22} = A_{2:n,2:n}$ is itself lower triangular and $(n-1)\times(n-1)$. Now $Ax = b$ is

$$\begin{pmatrix} a_{11} & 0_{1\times(n-1)} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_{2:n} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_{2:n} \end{pmatrix}$$

The top row of the matrix says $a_{11}x_1 = b_1$ so $x_1 = b_1/a_{11}$.

The bottom block of the matrix has $(n-1)$ rows

$$\begin{aligned}
\begin{pmatrix} A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_{2:n} \end{pmatrix} &= b_{2:n} \\
A_{21}x_1 + A_{22}x_{2:n} &= b_{2:n} \\
A_{22}x_{2:n} &= b_{2:n} - A_{21}x_1 \\
\tilde{A}\tilde{x} &= \tilde{b} \qquad \text{now } (n-1)\times(n-1)
\end{aligned}$$

We are left with a smaller version of the problem we started with.

It took $2(n-1)+1$ additions, subtractions, multiplications and divisions (called 'flops') to solve for $x_1$ and calculate $\tilde{A}$ and $\tilde{b}$. Since $\sum_{i=1}^{n}(2i-1) = n^2$, forward solving is $n^2$ flops.

R has `forwardsolve(A,b)` for forward elimination for $n \times n$ lower triangular `A` and $n \times 1$ `b`. There is `backsolve(A,b)` for backward elimination on upper triangular `A`.

## LU factorization

The most efficient method for solving $Ax = b$ for a general full rank $n \times n$ square matrix is to factorize

$$A = LU$$

into a lower $L$ and upper $U$ triangular matrices * at a cost of $2n^3/3 + O(n^2)$ flops (we havn't proven this, it's just assertion) and then solving $LUx = b$ by setting $y = Ux$ and then

$$\text{solving } Ly = b \text{ (forwards)}$$

and then

$$\text{solving } Ux = y \text{ (backwards).}$$

The function `solve(A,b)` uses this method. The two elimination steps take $2n^2$ flops so the leading term in the number of flops is $2n^3/3$.

*if there is no $LU$ factorization we seek $A = PLU$ with $P$ a permutation.

## Normal linear models

### Consider the aids data

```
> d = read.table("AIDS.txt")
> head(d)
  cases time time.sq
1   185    1       1
2   200    2       4
3   293    3       9
4   374    4      16
5   554    5      25
6   713    6      36
> (n<-dim(d)[1])
[1] 25
```

Suppose we want to fit the normal linear regression model

$$y_i = \alpha + \beta_1 x_i + \beta_2 x_i^2 + \varepsilon_i, \quad i = 1, 2, \ldots, n$$

with $y_i$ the number of cases in month $x_i$, and $\varepsilon_i \sim N(0, \sigma^2)$ iid normal errors. In vector form the model is

$$\begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & x_n & x_n^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_n \end{pmatrix}$$

or

$$y = X\theta + \varepsilon$$

with $\theta = (\alpha, \beta_1, \beta_2)^T$ etc.

The $R$ commands to fit this normal linear model are

`d.lm=lm(cases ~time+time.sq,data=d)`

`summary(d.lm)`

Here `d.lm` is a list full of results from the model fit output by `lm()`. Notice the R formula notation `cases~time+time.sq`.

The columns of `summary(d.lm)` output give $\hat{\theta}_i$, an estimate $\hat{\sigma}_i$ of the error in $\hat{\theta}_i$, and columns for the test H0: $\theta_i = 0$.

If the model is good, the regression should interpolate the data with normal residuals $y - X\hat{\theta}$. We can check this using a normal qq-plot for the residuals,
`qqnorm(residuals(d.lm)); qqline(residuals(d.lm))`.

## What's inside the `lm()` box?

The equations $X\theta = y$ are *over-determined* (more equations than variables, $n > p$, we cant expect a solution), so minimize $R(\theta) = (y - X\theta)^T(y - X\theta)$; get $X\theta$ as close as we can to $y$.

$$
\begin{aligned}
R(\theta) &= \sum_{i=1}^{n}(y_i - \alpha - \beta_1 x_i - \beta_2 x_i^2)^2 \\
&= (y - X\theta)^T(y - X\theta) \\
&= (X\theta)^T X\theta - 2y^T X\theta + y^T y
\end{aligned}
$$

Taking partial derivatives wrt $\theta$ and imposing $\frac{\partial R}{\partial \theta} = 0$ ($p$ equations) leads to the $p$ normal equations

$$X^T X\theta = X^T y$$

for $\theta$ in this over-determined system. This is $Ax = b$ with $A = X^T X$, $x = \theta$ and $b = X^T y$.

## Solving the normal equations using QR factorization

We could use LU factorization to solve the normal equations. However QR factorization is usually best as it is more stable numerically.

$$X = \begin{pmatrix} 1 & -1 \\ 0 & 10^{-10} \\ 0 & 0 \end{pmatrix} \qquad X^T X = \begin{pmatrix} 1 & -1 \\ -1 & 1 + 10^{-20} \end{pmatrix}$$

At machine precision $1 + 10^{-20}$ and $1$ are equal so $X^T X$ appears to be singular. Any method (like LU) that solves $(X^T X)\theta = X^T y$ by first computing $X^T X$ will fail on this problem.

Instead, factorize $X = QR$ ($Q$ is $n \times p$ and orthogonal, so $Q^T Q = I_{p \times p}$, and $R$ is $p \times p$, upper triangular, and has positive entries on the diagonal). This takes $2np^2$ flops (assertion). Since

$$X^T X = R^T Q^T Q R,$$

the normal equations

$$X^T X \theta = X^T y$$

are

$$R^T R \theta = R^T Q^T y.$$

We can solve these by

$$\text{solving } R\theta = Q^T y \text{ (backwards)}$$

($np + p^2$ flops) for an overall leading order cost of $2np^2$ flops. The functions `qr.solve(X,y)` and `lm()` use this method. LU would take $np^2$ but may fail.

In R,

```
X=cbind(rep(1,n),d$time,d$time.sq)
```

followed by

```
d.theta=qr.solve(X,d$cases)
```

to give the regression parameters.