Part A Simulation and Statistical Programming HT14

6th Practical: MCMC, convergence; Bayesian inference.

1. (a) Write an MCMC algorithm targeting Exp(μ) with mean $\mu = 1$. Check your MCMC.

   (b) Suppose we have an observation $X \sim Exp(\mu)$ and wish to estimate μ. If $X = x$ with $x = 1.7$ (say) and we have a Gamma($a = 3, b = 1$) prior for μ the posterior is

   $$\begin{aligned} \pi(\mu|x) &= \mu^{-1}\exp(x/\mu)\mu^{a-1}\exp(-b\mu) \\ &= \mu^{a-2}\exp(-b\mu - x/\mu) \end{aligned}$$

   Write an MCMC algorithm targeting $\pi(\mu|x)$ and use it to give an estimate the posterior mean for $\mu|x$.

2. In lecture 12 we gave an MCMC algorithm to sample a mixture of bivariate normals

   $$p(x) \propto 0.5N(x;\mu_1,\Sigma_1) + 0.5N(x;\mu_2,\Sigma_2)$$

   with $x = (x_1, x_2)$ etc.

```
a=3; n=2000
mu1=c(1,1); mu2=c(5,5); S=diag(2); S1i=S2i=solve(S);
X=matrix(NA,2,n); X[,1]=x=mu1
for (t in 1:(n-1)) {
   y<-x+(2*runif(2)-1)*a
   MHR<-f(y,mu1,mu2,S1i,S2i)/f(x,mu1,mu2,S1i,S2i)
   if (runif(1)<MHR) x<-y
   X[,t+1]<-x
}

#MCMC simulate X_t according to a mixture of normals
f<-function(x,mu1,mu2,S1i,S2i,p1=0.5) {
  #mixture of normals, density up to constant factor
  c1<-exp(-t(x-mu1)%*%S1i%*%(x-mu1))
  c2<-exp(-t(x-mu2)%*%S2i%*%(x-mu2))
  return(p1*c1+(1-p1)*c2)
}
```

   (a) Modify the MCMC so that the proposal is $y_i \sim N(x_i, a^2), i = 1, 2$.

   (b) Modify the mixture distribution so that it targets

   $$p(x) \propto (1/3)N(x;\mu_1,\Sigma_1) + (1/3)N(x;\mu_2,\Sigma_2) + (1/3)N(x;\mu_3,\Sigma_3)$$

   with $\mu_3 = (9, 9)^T$ and $\Sigma_3 = I_2$ (same as Σ_2 etc). Run your MCMC and make a scatter plot of the run in $\Re^2$.

(c) Run your MCMC for varying values of a such as $a = 0.1, 1, 10, 100$ plotting the MCMC trace of `X[1,]`, the first component.

(d) How would you define the "best" value of a?

3. A binary image has been corrupted by "salt and pepper" noise. If the true image was X, we observe Y where

$$Y_i = \begin{cases} X_i & \text{with probability } p \\ 1 - X_i & \text{otherwise} \end{cases}$$

Use the Ising model prior for X with smoothing parameter $\theta = 0.6$. Let $|X - Y|$ give the number of pixels disagreeing between X and Y. The likelihood for X is

$$L(X; Y) = p^{n^2 - |X-Y|}(1 - p)^{|X-Y|}.$$

Modify the MCMC code (which targets the posterior for the normal error model with Ising prior) from the lecture to target the posterior distribution

$$\pi(x|y) \propto p^{n^2 - |X-Y|}(1 - p)^{|X-Y|} \exp(-\theta \# x).$$

Simulate $X|\theta = 0.6$ using the Ising code from today and add S&P noise with $p = 0.8$ to generate Y. Estimate the posterior mean for $X|Y$.