1. (a) Consider two models with parameter spaces respectively $\theta \in \mathbb{R}^p$ and $\phi = (\theta, \psi) \in \mathbb{R}^{p+q}$. We want to compare model 1 with prior $\pi_1(\theta)$, observation model $p_1(y|\theta)$ and marginal likelihood $p_1(y)$ with model 2 where we have $\pi_2(\phi), p_2(y|\phi),$ and $p_2(y)$ correspondingly. Explain briefly why $\frac{p_1(y)}{p_2(y)} \neq \frac{E_{\phi|y,m=2}(\pi_1(\phi)p_1(y|\phi)h(\phi))}{E_{\theta|y,m=1}(\pi_2(\theta)p_2(y|\theta)h(\theta))}$.

(b) Let $Q(\psi)$ be a probability density on $\mathbb{R}^q$. Show that $\frac{p_1(y)}{p_2(y)} = \frac{E_{(\theta,\psi)|y,m=2}(Q(\psi)\pi_1(\theta)p_1(y|\theta)h(\theta, \psi))}{E_{\psi}(E_{\theta|y,m=1}(\pi_2(\theta, \psi)p_2(y|\theta, \psi)h(\theta, \psi)))}$.

where $\psi \sim Q$ in the expectation in the denominator and $h : \mathbb{R}^{p+q} \rightarrow \mathbb{R}$ is a function chosen so that the expectations exist. Comment briefly on how this last identity may be used for model comparison for models defined on spaces of unequal dimension.\(^1\)

(c) Briefly outline any assumptions we are making about the densities above.

2. Consider two urns. In the first urn there are 50 black balls and 50 red balls. In the second urn there are 100 balls, the number of each color unknown. Suppose the proportion of back balls in the second urn is equal $\phi$.

Jane’s $\phi$-prior, $\pi(\phi)$, satisfies $E(\phi) = 1/2$. Jane is offered a choice of urn and color and two balls are drawn (with replacement) from the chosen urn. Jane receives a £1 reward for each ball matching her chosen color. Her utility function is $U(0) = 0, U(1) = v, U(2) = 1$ with $1/2 < v < 1$.

Jane is offered red from the first urn or black from the second.

(a) Show that the expected utility of choosing the second urn given $\phi$ is $E(U|\phi) = 2\phi(1 - \phi)v + \phi^2$.

(b) Jane chooses the first urn. Show that this choice maximises the expected utility.

(c) Jane is now offered black from the first urn or red from the second. Show that Jane should again choose the first urn.

(d) In what sense does this suggest a resolution to the Ellsberg “paradox”.

3. The Savage axioms (as formulated by DeGroot) characterise coherent prior preference for events stated in terms of inequalities, so that $A \leq B$ implies $B$ is at least as likely as $A$.

(a) Write down the first three axioms (see Lecture notes).

(b) Show that preference relations satisfying these 3 axioms have the following properties:
   i. If \( A \leq B \) then \( A^c \geq B^c \) (where \( A^c \) is the complement of \( A \) etc);
   ii. (Optional) The order is transitive, ie, if \( A \leq B \) and \( B \leq C \) then \( A \leq C \).

4. For \( x_i \in \{0, 1\} \) for \( i = 1, 2, 3, \ldots \), the sequence of distributions \( p_n(x_1, \ldots, x_n), n = 1, 2, 3, \ldots \) is *marginally consistent* if

\[
p_n(x_1, \ldots, x_n) = p_{n+1}(x_1, \ldots, x_n, 0) + p_{n+1}(x_1, \ldots, x_n, 1).
\]

John elicits a separate prior \( p_n(x_1, x_2, \ldots, x_n) \) for each value of \( n = 1, 2, \ldots \). John’s priors are not marginally consistent, that is

\[
p_n(x_1, \ldots, x_n) \neq p_{n+1}(x_1, \ldots, x_n, 0) + p_{n+1}(x_1, \ldots, x_n, 1).
\]

(a) (optional) Can you think of a well-known family of distributions on binary random variables that is not marginally consistent? *Hint, a model for binary images...*

(b) Show that John’s priors do not satisfy the Savage axioms (consider the first three).

(c) Show that, under John’s priors, \( x_1, x_2, x_3, \ldots x_{n+1} \) is not part of an infinite exchangeable sequence. *Hint: show infinite exchangeable sequences are marginally consistent.*

5. Let \( x_1, x_2, x_3, \ldots \) be an infinite exchangeable sequence of binary random variables. Show that \( \text{cov}(x_i, x_j) \geq 0 \) for all \( i, j \in \{1, 2, 3, \ldots \} \).

6. Let \( X_1, X_2 \) be binary random variables. Table entries below give probabilities, \( p(x_1, x_2) = \Pr(X_1 = x_1, X_2 = x_2) \), for outcomes \((X_1, X_2) = (x_1, x_2)\) indicated by row and column.\(^2\)

\[
\begin{array}{c|cc}
X_2 & X_1 = 0 & X_1 = 1 \\
\hline
X_2 = 0 & 0 & 1/2 \\
X_2 = 1 & 1/2 & 0 \\
\end{array}
\]

(a) Show that \( X_1 \) and \( X_2 \) are exchangeable.

(b) Show that there does not exist a distribution \( F \) such that

\[
p(x_1, x_2) = \int_0^1 \prod_{i=1,2} p^{x_i}(1-p)^{1-x_i} \, dF(p),
\]

ie, de Finetti’s theorem need not hold if the exchangeable sequence is finite.

---