SC7/SM6 Bayes Methods

First problem Sheet for classes in week 2 for MSc students

1. For \( i = 1, \ldots, n \), let \( A_i \) be the event that student \( i \) enjoys the course. As far as we are concerned the students are exchangeable so our prior probability for each of the events \( A_1, \ldots, A_n \) is \( P(A_i) = p, \ i = 1, \ldots, n \). Subjective probabilities are manipulated in the same way as frequentist probabilities, so our prior on the proportion of successes in \( n \) trials should be approximately normal with mean \( p \) and variance \( p(1 − p)/n \). If \( n \) is large this prior expresses near certainty in the proportion of students enjoying the course. What has gone wrong with this prior elicitation? How might we fix this problem?

2. Suppose \( Y = y \) genuinely is a draw from the observation model \( p(y|θ) \), and nature truly drew \( θ \sim π(θ) \). Suppose \( y’ \sim p(y'|y) \) is a draw from the posterior predictive distribution (PPD). Show that \( y’ \) and \( y \) are exchangeable data and for \( i = 1, \ldots, N \) let \( s_i = s(y_{(i)}) \). Show that the rank of \( s(y) \) in the set \( s_{(1)} \ldots s_{(N)} \) is uniformly distributed in \( 1, 2, \ldots, N \). What is the connection to Goodness of Fit testing?

3. (a) Let \( X = (X_1, X_2) \) be a pair of random variables \( X \in \mathbb{R}^2 \) with joint density

   \[ p(x) \propto \exp(-|x_1| - |x_2| - |x_1 - x_2|), \quad \text{where } x = (x_1, x_2). \]

   Give an MCMC algorithm targeting \( p(x) \).

   (b) Let \( A = \{ x \in \mathbb{R}^2 : 2 < x_2 < 2.01 \} \). Write down an MCMC algorithm targeting \( p(x|X \in A) \) giving the acceptance probability as an explicit function of \( x_1, x_2 \). In choosing the proposal distribution you should consider the efficiency of your algorithm.

   (c) Implement your algorithm for the last part in R and carry out convergence checks.

4. Let \( Γ(x; α, β) \) be the Gamma density. Consider Poisson observations \( Y = (Y_1, Y_2, \ldots, Y_n) \) with means \( λ = (λ_1, λ_2, \ldots, λ_n) \) given by a mixture of Gamma densities: for shape parameters \( α_1, α_2 \) and rate parameters \( β_1, β_2 \), a known mixture proportion \( 0 < p < 1 \) and \( i = 1, 2, \ldots, n \), we observe

   \[ Y_i|λ_i \sim \text{Poisson}(λ_i) \]

   (all iid) with

   \[ λ_i \sim pΓ(λ_i; α_1, β_1) + (1 − p)Γ(λ_i; α_2, β_2). \]

   (a) Denote by \( π(α_1, β_1, α_2, β_2) \) a prior for the unknown shape and rate parameters. Write down the joint posterior for \( α_1, β_1, α_2, β_2 \) and \( λ \) given \( Y_1, Y_2, \ldots, Y_n \). Give an MCMC algorithm sampling \( α_1, β_1, α_2, β_2, λ|Y_1, \ldots, Y_n \).

   (b) Integrate \( λ \) out of the joint posterior to obtain a marginal posterior density for \( α_1, β_1, α_2, β_2|Y_1, \ldots, Y_n \). Comment briefly on how you would alter your MCMC algorithm for the new target. What considerations would guide your choice of simulation method (ie, whether to simulate the joint or the marginal posterior density)?

5. (optional) A contingency table \( X \) is an \( n \times m \) matrix with non-negative integer entries \( X_{i,j} \geq 0 \) and fixed row sums \( r = (r_1, r_2, \ldots, r_n) \) and fixed column sums \( c = (c_1, c_2, \ldots, c_m) \) (which we suppose are known). Suppose we have contingency table data \( Y \) gathered by observing each entry in the contingency table independently with a Poisson observation process so that \( Y_{i,j} \sim \text{Poisson}(X_{i,j}) \).

   (a) Give a Bayesian analysis leading to a Posterior distribution for \( X|Y, r, c \).

   (b) Give an MCMC algorithm simulating \( X|Y, r, c \) and show that it is ergodic. Hint: Suppose \( Z \) is an \( n \times m \) matrix with all entries zero except four entries located at the corners of a rectangle, \( Z_{i,j} = Z_{a,b} = 1 \) and \( Z_{a,j} = Z_{i,b} = −1 \) (with \( i \neq a \) and \( j \neq b \)). If we generate a random \( Z \) (by choosing \( i \neq a \) and \( j \neq b \) at random) and set \( X' = X + Z \) then \( X' \) and \( X \) have the same row and column sums.