SC7/SM6 Bayes Methods HT20

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Lecture 12: Reversible Jump MCMC II.*

Notes and Problem sheets are available at

http://www.stats.ox.ac.uk/~nicholls/BayesMethods/

Reversible Jump MCMC

Consider MCMC targeting our model-averaging posterior

\[ \pi(\theta, m|y) \propto p(y|\theta, m)\pi(\theta|m)\pi(m), \quad (\theta, m) \in \Omega \]

with \( m \in \mathcal{M} \), and given \( m, \theta \in \Omega_m \) with \( \theta = (\theta_1, ..., \theta_m) \) and

\[ \Omega = \bigcup_{m \in \mathcal{M}} \bigcup_{\theta \in \Omega_m} \{(\theta, m)\}. \]

The MCMC state is \( X_t = (\theta, m) \) for some state \( (\theta, m) \in \Omega \).

Suppose we target \( \pi(\theta, m|y) \) with \( \theta_i \in \mathbb{R}^k, \quad i = 1, ..., m, \) for \( k \) a fixed positive integer, and have \( \Omega_m = \mathbb{R}^{km} \) so \( \operatorname{dim}(\theta) = km \).

When we move between models the dimension of \( \theta \) changes.
RJ Updates:

Let $\rho_{m,m'}$ give the probability to propose a move to model $m'$ given the current model is $m$. In the following $\rho_{m,m'} = 0$ unless $|m - m'| = 1$ (add or delete k-dimensions at a time).

Consider an update adding an entry to the vector so $m' = m + 1$. The generating density for $(\theta, m) \rightarrow (\theta', m')$ is $g_{m,m'}(u), \ u \in U_{m,m'}$. Here we must have $\dim(u) = k$.

Let $u \sim g_{m,m'}$ and $\theta' = \psi_1(\theta, u)$ with

$$\theta' = (\theta_1, ..., \theta_m, \theta_{m+1}).$$

Drop $m$ from $(\theta, m, u)$ as $m = |\theta|$, the number of elements.

We choose the reverse move $(\theta', m') \rightarrow (\theta, m)$ with probability $\rho_{m',m}$. Let $U_{m',m} = \{\emptyset\}$ and $g_{m',m}(\emptyset) = 1$. 
For \((\theta, u) \in \Omega_m \times \mathcal{U}_{m,m'} \) and \((\theta', u') \in \Omega_{m'} \times \mathcal{U}_{m',m} \) let
\[(\theta', u') = \psi(\theta, u)\]
with
\[\psi(\theta, u) = (\psi_1(\theta, u), \psi_2(\theta, u))\]
the invertible mapping for this update-pair. We require
\[(\theta', \emptyset) = (\psi_1(\theta, u), \psi_2(\theta, u)), \theta' \in \Omega_{m'}\]
and
\[(\theta, u) = (\psi_1(\theta', \emptyset), \psi_2(\theta', \emptyset)).\]
Here \(u\) solves \(\theta'_{m+1} = \psi_1(\theta, u)_{m+1}\). This determines \(\psi_2(\theta', \emptyset)\).

\(\psi\) is a differentiable involution on \(\Omega_m \times \mathcal{U}_{m,m'} \cup \Omega_{m'} \times \mathcal{U}_{m',m} \).
Claim: the acceptance probability for \((\theta, m) \to (\theta', m')\) is
\[
\alpha(\theta', m'|\theta, m) = \min \left\{ 1, \frac{\pi(\theta', m'| y)q(\theta, m|\theta', m')}{\pi(\theta, m|y)q(\theta', m'|\theta, m)} \right\},
\]
with \(q(\theta', m'|\theta, m)\) the “proposal generating distribution”
\[
q(\theta', b|\theta, a) = \begin{cases} 
\rho_{a,b}g_{a,b}(u) \left| \frac{\partial \theta'_{a+1}}{\partial u} \right|^{-1} & \text{if } b = a + 1 \\
\rho_{a,b}I_{\theta'=(\theta_1, \ldots, \theta_{a-1})} & \text{if } b = a - 1
\end{cases}
\]

Remark: \((*)\) is the usual form. If \(J_\psi\) is non-singular we write it straight down and just calculate our \(q(\theta', m'|\theta, m)\).

Proof: Quoting the expression from the end of the last lecture, the acceptance probability for \((\theta, m) \to (\theta', m')\) is
\[
\alpha(\theta', m'|\theta, m) = \min \left\{ 1, \frac{\pi(\theta', m'| y)\rho_{m', m}g_{m', m}(\emptyset)}{\pi(\theta, m|y)\rho_{m, m'}g_{m, m'}(u)}J_\psi(\theta, u) \right\}
\]
with

\[ J_\psi(\theta, u) = \begin{vmatrix} \frac{\partial \theta'_1:m}{\partial \theta} & \frac{\partial \theta'_1:m}{\partial u} \\ \frac{\partial \theta'_{m+1}}{\partial \theta} & \frac{\partial \theta'_{m+1}}{\partial u} \end{vmatrix} \]

\[ = \begin{vmatrix} \frac{\partial \theta'_{m+1}}{\partial u} \end{vmatrix}, \]

since \( \frac{\partial \theta'_{1:m}}{\partial u} = 0_{m\times k} \) and \( \frac{\partial \theta'_{1:m}}{\partial \theta} = I_{km\times km} \). But then

\[ \alpha(\theta', m'|\theta, m) = \min \left\{ 1, \frac{\pi(\theta', m'|y)\rho_{m',m}}{\pi(\theta, m|y)\rho_{m,m'}g_{m,m'}(u)J_{\psi}^{-1}} \right\} \]

is given by the expression in the claim. [EOP]

Exercise: for the deletion update \((\theta', m') \rightarrow (\theta, m)\) verify

\[ \alpha(\theta, m|\theta', m') = \min \left\{ 1, \frac{\pi(\theta, m)q(\theta', m'|\theta, m)}{\pi(\theta', m')q(\theta, m|\theta', m')} \right\}. \]
RJMCMC - simple example

Consider a random variable taking values $X = 1/2$ with probability 1/3 and otherwise $X \sim 2x\mathbb{I}_{0<x<1}$.*

If $\pi(x, m) = \pi(x|m)\pi(m)$ with $\pi(m = 1) = 1/3$, $\pi(m = 2) = 2/3$, $\pi(x|m = 1) = \mathbb{I}_{x=1/2}$, $\pi(x|m = 2) = 2x$, then the process above is the same as $m \sim \pi(m)$ and then $x \sim \pi(x|m)$.

Here $\Omega_1 = \{1/2\}, \Omega_2 = (0, 1)$ and $(x, m) \in \Omega$ with

$$\Omega = \{(1/2, 1)\} \cup \bigcup_{x \in (0,1)} \{(x, 2)\}$$

and

$$\pi(x, m) = \begin{cases} 
1/3 & \text{if } (x, m) = (1/2, 1), \\
4x/3 & \text{if } m = 2 \text{ and } x \in (0, 1), \text{ and} \\
0 & \text{otherwise.}
\end{cases}$$

*The CDF is $F_X(x) = \Pr(X \leq x)$ with $F_X(x) = \frac{2}{3}x^2 + \frac{1}{3}\mathbb{I}_{x\geq1/2}$. 
**RJMCMC algorithm targeting** \((X, M) \sim \pi(x, m)\):

MCMC state: \((X_t, M_t) = (x, m)\). Suppose \((x, m) = (1/2, 1)\).

Increase dimension, \((1/2, 1) \rightarrow (x', 2)\): if \(m = 1\) propose \(m' = 2\) with probability \(\rho_{1,2} = 1\) and simulate \(x' \sim q(x')\).

In notation above \(u \sim g_{1,2}(u)\) with \(x' = u\) and \(g_{1,2}(x') = q(x')\). We set \(x' = \psi_1(x, u) = u\) and \(u' = \psi_2(x, u) = \emptyset\).

Choose \(q(x')\) anything irreducible. Since \(\Omega_2 = (0, 1)\), I use

\[q(x') = \text{Beta}(x'; \alpha = 1/2, \beta = 1/2) \quad (!)\]

to emphasise that any irreducible choice works.

Decrease dimension \((x', 2) \rightarrow (1/2, 1)\): if \(m' = 2\) propose \(m = 1\) with probability \(\rho_{2,1} = 1\). Set \(x = 1/2\) and \(u = x'\).

This is \(u' \sim g_{2,1}\) with \(g_{2,1}(\emptyset) = 1\) so \(u' = \emptyset\) and we set \(x = \psi_1(x', \emptyset) = 1/2\) and \(u = \psi_2(x', \emptyset) = x'\).
Acceptance probabilities:

Increase dimension, \((1/2, 1) \rightarrow (x', 2)\): acceptance probability,

\[
\alpha(x', m'|x, m) = \min \left\{ 1, \frac{\pi(x', m') \rho_{m',m} g_{2,1}(u')}{\pi(x, m) \rho_{m,m'} g_{1,2}(u)} J_\psi(x, u) \right\}.
\]

We set \((x', u') = \psi(x, u)\) with \(\psi(x, u) = (u, \emptyset)\) so \(J_\psi(x, u) = |\partial x'/\partial u| = 1\). Substituting \(g_{2,1} = 1, g_{1,2}(u) = q(x')\) etc,

\[
\alpha(x', m'|x, m) = \min \left\{ 1, \frac{4x'/3}{\text{Beta}(x'; \alpha, \beta)/3} \right\}.
\]

Decrease dimension \((x', 2) \rightarrow (1/2, 1)\): as an exercise verify

\[
\alpha(x, m|x', m') = \min \left\{ 1, \frac{\text{Beta}(x'; \alpha, \beta)/3}{4x'/3} \right\}.
\]

Iteration: we generate our chain \((X_t, M_t)\) iterating proposals and acceptance steps using the formula above.
Remark 1: the dimension of the proposal dbn matches the change in dimension in the target state

$$\dim(\Omega_1 \times \mathcal{U}_{1,2}) = \dim(\Omega_2 \times \mathcal{U}_{2,1})$$

since $$\dim(\{1/2\} \times (0, 1)) = \dim((0, 1) \times \{\emptyset\}) = 1.$$  

Remark 2: notice that

$$\alpha(x', m'| x, m) = \min \left\{ 1, \frac{\pi(\theta', m')q(x, m|x', m')}{\pi(x, m)q(x', m'| x, m)} \right\},$$

with $$q(x, m|x', m') = \mathbb{I}(x, m) = (1/2, 1)$$ and $$q(x', m'| x, m) = q(x').$$ In this example, these are simply the probabilities (pmf, pdf) to generate the candidate state.

Remark 3: a fixed-dimension update sets $$\rho_{2,1} = \rho_{2,2} = 1/2$$ so we have two options if $$m = 2$$. The update $$(x, 2) \rightarrow (x', 2)$$ would target $$\pi(x|m = 2)$$ using standard MCMC tools.