1. Consider a random sample of size n from a distribution with probability density function $f(x;\theta)$, and consider a test of the null hypothesis $H_0 : \theta = \theta_0$ against the alternative $H_1 : \theta = \theta_1$. Define the terms *type I error*, and *type II error*. State and prove the Neyman-Pearson lemma.

Let $X_1, \ldots, X_n$ be independent $N(\theta, \sigma_0^2)$ random variables, where σ_0^2 is known. Find the best critical region for a test of size α of $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, where $\theta_1 > \theta_0$.

Show that the power function $Q(\theta)$ of this test is given by

$$Q(\theta) = 1 - \Phi\left(\frac{\sqrt{n}}{\sigma_0}(\theta_0 - \theta) + z_\alpha\right)$$

where Φ is the cumulative distribution function of the standard normal distribution and $\Phi(z_\alpha) = 1 - \alpha$.

If $\theta_0 = 0$, $\theta_1 = 0.5$ and $\sigma_0 = 1$, how large must n be if $\alpha = 0.05$ and the power at θ_1 is to be 0.975?

2. A telephone receptionist for a large partnership of financial advisers is responsible for determining the precise nature of each incoming enquiry and connecting the client with an appropriate adviser. The number of inappropriate connections on any given day may be modelled by a random variable X which has a Poisson distribution with mean μ. If Z is the number of inappropriate connections made over a period of n days, determine the distribution of Z and find its expected value.

Uhura, who has been such a receptionist for many years, has been found to have a mean rate of $\mu_U = 0.47$ inappropriate connections per day. For several months she has been training Spock, a new receptionist, with corresponding mean rate μ_S. At a meeting of senior partners, it was conjectured that Spock was already as proficient as Uhura; accordingly they resolved to keep a daily record of the number of inappropriate connections made by him over his next 10 working days. Find a critical region of size 5% for a test of the hypothesis that Spock is as proficient as Uhura versus the alternative that he is less proficient.

For what values of μ_S does the probability of type II error fall below 10%?

{Note that, if $\varphi_\mu(k) = \sum_{x=0}^{k} \mu^x e^{-\mu} / x!$, then $\varphi_{4.7}(8) = 0.95$, $\varphi_{13}(8) = 0.1$.}

3. Explain how the likelihood ratio test can be used to carry out a test of a hypothesis H_0 against an alternative H_1 (i) when H_0 and H_1 are simple hypotheses, (ii) when both are composite, with $H_0 \subset H_1$. In case (ii) for large samples, which statistic is used to calculate the p-value of the test? What is its asymptotic distribution?

In studying the sex ratio in a population using a sample of size n, it is usually assumed that each child is male with probability p, independently of all other children. In such a study Renkonen (1956) observed 19,711 male births out of a total of 38,562 births in American families with two children each. Use the likelihood ratio statistic to test the hypothesis $H_0 : p = \frac{1}{2}$ against a suitable alternative, which you should specify.

Renkonen also found 17,703 males out of 35,042 similar births in Finland. Use the generalised likelihood ratio test to test the hypothesis that p has the same value in each country versus a suitable alternative.

4. A random variable X has a distribution given by

$$P(X = i) = \pi_i \quad (i = 1, 2, \ldots, p),$$

where $\sum_{i=1}^{p} \pi_i = 1$. In a sample of size n from a population with distribution X, the frequency of outcome i is n_i, where $n_i \geq 0$ and $\sum_{i=1}^{p} n_i = n$. Find the maximum likelihood estimators of π_i.

The seeds of the Indian creeper plant *Pharbitis nil* are valued for their purgative properties. The leaves can be variegated or unvariegated and, at the same time, faded or unfaded. In an experiment reported by Bailey (1961), plants were crossed in such a way that their offspring would produce variegated leaves with probability $\frac{1}{4}$ and faded leaves with probability $\frac{1}{4}$. What probability model would you propose under the assumption that the properties of variegated appearance and faded appearance are independent?

Of 290 plants which were observed, 31 had variegated faded leaves, 37 had variegated unfaded leaves, 35 had unvariegated faded leaves and 187 had unvariegated unfaded leaves. Use a χ^2 test of goodness-of-fit to show that the data offer strong evidence that your independence model is inappropriate.

A genetic theory which allows for an effect called *genetic linkage* assumes a probability model for the above observations with respective probabilities

$$\frac{1}{16} + \theta, \frac{3}{16} - \theta, \frac{3}{16} - \theta, \frac{9}{16} + \theta.$$

Find $\widehat{\theta}$, the maximum likelihood estimate of θ.

Let H_0 be the null hypothesis that the genetic linkage model is appropriate with general alternative H_1. If L_0 is the supremum of the likelihood under H_0 and if L_1 is the supremum of the likelihood under H_1, show that

$$\Lambda = 2 \sum_{i=1}^{4} n_i \log \left(\frac{n_i}{n\pi_i\left(\widehat{\theta}\right)} \right),$$

where $\Lambda = -2(\log L_0 - \log L_1)$. Write down the approximate distribution of Λ.

What can you infer about the plausibility of the genetic linkage model?

[You may remember that you met the genetic linkage model in Problem Sheet 1, Question 6. You might also like to try the test on those data.]

5. The ordered pairs of random variables (X_k, Y_k), $k = 1, 2, \ldots, N$, are independent and

$$P((X_k, Y_k) = (i, j)) = \theta_{ij}, \quad i = 1, 2, \ldots, r;\ j = 1, 2, \ldots, c,$$

where $\sum_{i,j} \theta_{ij} = 1$. The frequency of the outcome (i, j) is n_{ij}, where $n_{ij} > 0$.

Find the maximum likelihood estimators for θ_{ij} assuming that

(i) $\theta_{ij} = \phi_i \rho_j, \quad i = 1, 2, \ldots, r;\ j = 1, 2, \ldots, c$, where $\sum_i \phi_i = \sum_j \rho_j = 1$, and

(ii) without this assumption.

Hence find test statistics for the null hypothesis that the X_ks and Y_ks are independent using

(a) the likelihood ratio method,

(b) Pearson's χ^2 statistic.

What can you say about the distributions of the two statistics for large values of n?

Some individuals are carriers of the bacterium *streptococcus pyogenes*. To investigate whether there is a relationship between carrier status and tonsil size in schoolchildren, 1398 children were examined and classified according to their carrier status and tonsil size. The data appear below. Is there an association?

Tonsil size	Carrier status	
	Carrier	Non-carrier
Normal	19	497
Large	29	560
Very large	24	269