Mods Statistics, Sheet 3, HT 2012

1. Construct a central $100(1-\alpha)\%$ confidence interval for the unknown parameter μ based on a random sample of size n from a normal distribution with mean μ and variance 1.

If $\alpha = 0.05$ and the length of the interval is to be less than 1, how large must n be? What if the length is to be less than 0.1?

2. Suppose $X_1, \ldots, X_n$ is a random sample from a Bernoulli distribution with probability $P(X_i = 1) = p$.

(i) What is the variance of X_i? Show that the estimator $\widehat{p} = \overline{X}$ has expectation p and find its variance.

(ii) Using the central limit theorem construct a random variable which has an approximate standard normal distribution and indicate how this can be used to find a $100(1-\alpha)\%$ confidence interval for p.

(iii) Fifty female black ducks from locations in New Jersey were captured and radio-tagged prior to severe winter months. Of these 19 died during the winter. Find a 95% confidence interval for the proportion surviving. Claims made by environmentalists suggested a 50-50 chance of survival. Is this reasonable?

3. Suppose $X_1, \ldots, X_n$ are independent Poisson random variables each with mean θ. Assuming n is large and using the central limit theorem, construct (i) a central confidence interval for θ, and (ii) an upper confidence limit for θ, each with an associated confidence of $1-\alpha$.

4. Suppose $X_1, \ldots, X_n$ is a random sample from a Pareto distribution with probability density function

$$f(x; \lambda) = \lambda x^{-(\lambda+1)} \quad \text{for } x \geqslant 1,$$

where $\lambda > 2$. Find the mean μ of this distribution, and explain why the sample mean $\overline{X}$ might be suggested as an estimator of μ. Find the variance of the estimator $\overline{X}$.

Derive the following approximate 95% confidence interval for μ when the sample size is large

$$\left(\overline{x} - 1.96\sqrt{\frac{\overline{x}(\overline{x}-1)^2}{n(2-\overline{x})}},\ \overline{x} + 1.96\sqrt{\frac{\overline{x}(\overline{x}-1)^2}{n(2-\overline{x})}}\right).$$

[In questions 2, 3, and 4, respectively, first assume that it is permissible to replace p, θ, μ, by $\overline{x}$ to obtain an estimate of the variance of $\widehat{p}$, $\widehat{\theta}$, $\widehat{\mu}$. For 2 and 3, can you find a confidence interval without doing this?]