Mods Statistics, Sheet 2, HT 2012

1. If $X_1, \ldots, X_n$ is a random sample from a geometric distribution with parameter p, find the maximum likelihood estimator $\widehat{p}$ of p.

Let $\theta = 1/p$. Find the likelihood as a function of θ, the maximum likelihood estimator $\widehat{\theta}$, and verify that $\widehat{\theta} = 1/\widehat{p}$.

Show that $\widehat{\theta}$ is unbiased. In the case $n = 1$ show that $E(\widehat{p}) > p$.

2. Suppose $X_1, \ldots, X_n$ is a random sample from a $N(\mu, \sigma^2)$ distribution, where $\mu = \sigma^2 = \theta$. Show that the maximum likelihood estimator of θ is

$$\widehat{\theta} = \frac{1}{2}\left\{\left(1 + \frac{4}{n}\sum_{j=1}^{n} X_j^2\right)^{1/2} - 1\right\}.$$

3. The number of organisms in volume v of a liquid has a Poisson distribution with mean ρv, where ρ is the unknown density of organisms per unit volume. To estimate ρ an experimenter takes independent samples of liquid of volumes $v_1, \ldots, v_n$. Find the maximum likelihood estimator $\widehat{\rho}$ of ρ. Show that $\widehat{\rho}$ is unbiased and that its variance is the same for all choices of $v_1, \ldots, v_n$ with $\sum v_i = V$ (where V is fixed).

4. Suppose $X_1, \ldots, X_n$ is a random sample from a distribution with probability density function

$$f(x;\theta) = \begin{cases} e^{-(x-\theta)} & \text{if } x \geqslant \theta, \\ 0 & \text{otherwise.} \end{cases}$$

Find the maximum likelihood estimator of θ.

5. The following data (from Dyer (1981)) are annual wages (in multiples of 100 US dollars) of a random sample of 30 production line workers in a large American industrial firm.

Annual wages (hundreds of US $)

112	154	119	108	112	156	123	103	115	107
125	119	128	132	107	151	103	104	116	140
108	105	158	104	119	111	101	157	112	115

A standard probability model used for data on wages is the Pareto distribution, which has probability density function

$$f(x;\theta) = \theta\alpha^{\theta}x^{-(\theta+1)} \quad \text{for } x \geqslant \alpha,$$

where $\theta > 0$ and the constant α represents a statutory minimum wage. Find the maximum likelihood estimator of θ from a random sample $X_1, X_2, \ldots, X_n$, and, assuming $\alpha = 100$, the maximum likelihood estimate for the above dataset (for which $\sum \log x_i = 143.5$).

Now suppose there is no statutory minimum wage, so that α is also an unknown parameter. Show that the mle for α is $\widehat{\alpha} = \min_i X_i$.

Hence show that

$$P(\widehat{\alpha} > y) = \left(\frac{\alpha}{y}\right)^{n\theta},$$

(use the fact that $\min_i X_i > y$ iff $X_i > y$ for $i = 1, 2, \ldots, n$) and thus that for $\varepsilon > 0$,

$$P(|\widehat{\alpha} - \alpha| > \varepsilon) \to 0 \quad \text{as } n \to \infty.$$