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Increasing sets and product measures

Let $\mathbf{B} = (B_i, i \in I)$ be a collection of random variables taking values 0 and 1, where I is some arbitrary index set. Write P_x for the measure under which the (B_i) are i.i.d. with $P_x(B_i = 1) = x$.

We consider increasing events A depending on the collection $\mathbf{B}$ (and write $A(\mathbf{B})$ when we need to make the dependence explicit).

Theorem 1 *Let A be any increasing event. Suppose $k \in \mathbb{N}$, $\epsilon > 0$ and $0 < x^- < x^+ < 1$ are such that*

$$x^+ \geq 1 - (1 - x^-)^{1+1/k} \tag{1}$$

and

$$P_{x^-}(A) \in (k\epsilon^{1/2}, 1 - k\epsilon^{1/2}). \tag{2}$$

Then

$$P_{x^+}(A) - P_{x^-}(A) \geq \epsilon. \tag{3}$$

[N.B.: by considering simple symmetries one gets the same result under the conditions that $(x^+)^{1+1/k} \geq x^-$ and $P_{x^+}(A) \in (k\epsilon^{1/2}, 1 - k\epsilon^{1/2})$.]

Proof: Extend the space so that as well as $\mathbf{B}$ it now contains $k+1$ further independent copies of the collection of Bernoulli random variables, which we call $\mathbf{B}^{(1)}, \mathbf{B}^{(2)}, \dots, \mathbf{B}^{(k+1)}$. Under P_x, all the Bernoulli random variables share the same success probability x.

Write $x_{k+1} = 1 - (1 - x)^{k+1}$ and $x_k = 1 - (1 - x)^k$.

If we take the componentwise max $\mathbf{B}^{(1)} \vee \mathbf{B}^{(2)} \cdots \vee \mathbf{B}^{(k)}$, this gives a collection of independent Bernoulli random variables which, under P_x, have success probability x_k.

Similarly taking the max $\mathbf{B}^{(1)} \vee \mathbf{B}^{(2)} \cdots \vee \mathbf{B}^{(k+1)}$ leads to a collection of independent Bernoulli random variables with success probability x_{k+1}.

We write

$$A_k = A(\mathbf{B}^{(1)} \vee \mathbf{B}^{(2)} \cdots \vee \mathbf{B}^{(k)}),$$
$$A_{k+1} = A(\mathbf{B}^{(1)} \vee \mathbf{B}^{(2)} \cdots \vee \mathbf{B}^{(k+1)}).$$

Thus from the previous two paragraphs,

$$P_{x_k}(A(\mathbf{B})) = P_x(A_k),$$
$$P_{x_{k+1}}(A(\mathbf{B})) = P_x(A_{k+1}).$$

Since $A_k \subseteq A_{k+1}$, we have $P_{x_{k+1}}(A(\mathbf{B})) - P_{x_k}(A(\mathbf{B})) = P_x(A_{k+1} \setminus A_k)$.

We define $Y_r = P_x(A_k | \mathbf{B}^{(1)}, \mathbf{B}^{(2)}, \dots \mathbf{B}^{(r)})$.

Thus $Y_0 = P_x(A_k)$ a.s. and $Y_k = I(A_k)$ a.s.

We will show that if $P_x(A_{k+1} \setminus A_k)$ is small, then all the differences $Y_r - Y_{r-1}$ are likely to be small. This will then imply that $P_x(A_k)$ must be close to 0 or 1.

We make the following claim:

$$\text{if } P_x(Y_r - Y_{r-1} \geq y) \geq z, \text{ then } P_x(A_{k+1} \setminus A_k) \geq yz. \tag{4}$$

We postpone the proof of (4). First we show how it implies the main result.

Given ϵ, we will use $y = z = \epsilon^{1/2}$ in (4).

Suppose that $P_x(A_{k+1} \setminus A_k) < yz = \epsilon$. Then by (4), for each r we have $P_x(Y_r - Y_{r-1} \geq \epsilon^{1/2}) < \epsilon^{1/2}$.

In that case $P_x(Y_k - Y_0 \geq k\epsilon^{1/2}) < k\epsilon^{1/2}$.

That is, $P_x(I(A_k) - P_x(A_k) \geq k\epsilon^{1/2}) < k\epsilon^{1/2}$.

This means that either $P_x(A_k) \geq 1 - k\epsilon^{1/2}$ or $P_x(A_k) < k\epsilon^{1/2}$.

We have deduced that if $P_x(A_k) \in (k\epsilon^{1/2}, 1 - k\epsilon^{1/2})$, then $P_x(A_{k+1} \setminus A_k) \geq \epsilon$.

Translating, we obtain that if $P_{x_k}(A) \in (k\epsilon^{1/2}, 1 - k\epsilon^{1/2})$, then $P_{x_{k+1}}(A) - P_{x_k}(A) \geq \epsilon$.

To complete the proof, we choose x in such a way that

$$x^- = 1 - (1-x)^k = x_k$$

and

$$x^+ \geq 1 - (1-x)^{k+1} = x_{k+1}.$$

This is possible because of the assumption (1).

Then $P_{x^-}(A) = P_{x_k}(A)$ and $P_{x^+}(A) \geq P_{x_{k+1}}(A)$ and the desired result follows.

It remains to prove the claim (4), which we write as a separate lemma:

Lemma 1 *If $P_x(Y_r - Y_{r-1} \geq y) \geq z$, then $P_x(A_{k+1} \setminus A_k) \geq yz$.*

Proof of lemma:

Y_r is a function of $\mathbf{B}^{(1)}, \ldots, \mathbf{B}^{(r)}$.

We will consider an altered version of Y_r where $\mathbf{B}^{(r)}$ is replaced by $\mathbf{B}^{(k+1)}$.

Namely, define

$$\tilde{A}_k^r = A(\mathbf{B}^{(1)}, \ldots, \mathbf{B}^{(r-1)}, \mathbf{B}^{(r+1)}, \ldots, \mathbf{B}^{(k+1)}),$$

where $\mathbf{B}^r$ is omitted, and define

$$\tilde{Y}^r = P_x(\tilde{A}_k^r | \mathbf{B}^{(1)}, \ldots, \mathbf{B}^{(r-1)}, \mathbf{B}^{(k+1)}).$$

Now, conditional on Y_{r-1}, the quantities Y_r and $\tilde{Y}_r$ have the same distribution. In particular, $Y_r - Y_{r-1}$ and $\tilde{Y}_r - Y_{r-1}$ have the same distribution.

We expand:

$$\begin{aligned}
\tilde{Y}_r - Y_{r-1} &= P_x(\tilde{A}_k^r | \mathbf{B}^{(1)}, \mathbf{B}^{(2)}, \ldots, \mathbf{B}^{(r-1)}, \mathbf{B}^{(k+1)}) - P_x(A_k | \mathbf{B}^{(1)}, \mathbf{B}^{(2)}, \ldots, \mathbf{B}^{(r-1)}) \\
&= P_x(\tilde{A}_k^r | \mathbf{B}^{(1)}, \mathbf{B}^{(2)}, \ldots, \mathbf{B}^{(r-1)}, \mathbf{B}^{(k+1)}) - P_x(A_k | \mathbf{B}^{(1)}, \mathbf{B}^{(2)}, \ldots, \mathbf{B}^{(r-1)}, \mathbf{B}^{(k+1)}) \\
&\leq P_x(A_{k+1} | \mathbf{B}^{(1)}, \mathbf{B}^{(2)}, \ldots, \mathbf{B}^{(r-1)}, \mathbf{B}^{(k+1)}) - P_x(A_k | \mathbf{B}^{(1)}, \mathbf{B}^{(2)}, \ldots, \mathbf{B}^{(r-1)}, \mathbf{B}^{(k+1)}) \\
&= P_x(A_{k+1} \setminus A_k | \mathbf{B}^{(1)}, \mathbf{B}^{(2)}, \ldots, \mathbf{B}^{(r-1)}, \mathbf{B}^{(k+1)}),
\end{aligned}$$

where we have used variously that A_k and $\mathbf{B}^{(1)}, \ldots, \mathbf{B}^{(r-1)}$ are independent of $\mathbf{B}^{(k+1)}$, that $\tilde{A}_k^r \subseteq A_{k+1}$, and that $A_k \subseteq A_{k+1}$.

As a result we have that if $P_x(Y_r - Y_{r-1} \geq y) \geq z$ then $P_x(\tilde{Y}_r - Y_{r-1} \geq y) \geq z$, so that also

$$P_x(P_x(A_{k+1} \setminus A_k | \mathbf{B}^{(1)}, \mathbf{B}^{(2)}, \ldots, \mathbf{B}^{(r-1)}, \mathbf{B}^{(k+1)}) \geq y) \geq z,$$

which implies that $P_x(A_{k+1} \setminus A_k) \geq yz$ as desired. □