**Problem Sheet 4 - Part B Actuarial Science II - Oxford HT 2006**

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1. Let $\mu_x$ be the force of mortality, and $\ell_x$ be the corresponding life table.

   (i) Show that ${}_{n|m}q_x = \int_n^{n+m} {}_tp_x \mu_{x+t} dt$
   *[Here ${}_{n|m}q_x$ means the probability of death of a life currently aged $x$ between times $x+n$ and $x+n+m$.]*

   (ii) Show that $\mu_{x+0.5} \approx -\log p_x$ and $\mu_x \approx -0.5\,(\log p_x + \log p_{x-1})$.

   (iii) If $\ell_x = 100(100-x)^{1/2}$, find $\mu_{84}$ exactly.

2. (i) Gompertz's Law has $\mu_x^{(1)} = Bc^x$. Show that the corresponding survival function is given by ${}_tp_x^{(1)} = g^{c^x(c^t-1)}$ where $\log g = -B/\log c$.

   (ii) Makeham's Law has $\mu_x^{(2)} = A + Bc^x$. Show that ${}_tp_x^{(2)} = s^t\,{}_tp_x^{(1)}$ where $s = e^{-A}$.

   (iii) If $\mu_x = A\log x$, find an expression for $\ell_x/\ell_0$.

   (iv) If it is assumed that A1967-70 table follows Makeham's Law, use $\ell_{30}$, $\ell_{40}$, $\ell_{50}$ and $\ell_{60}$ to find $A$, $B$ and $c$.
   ($\ell_{30} = 33839$, $\ell_{40} = 33542$, and $\ell_{50} = 32670$, $\ell_{60} = 30040$).

3. The force of mortality for table 2 has twice the force of mortality for table 1.

   (i) Show that the probability of survival for $n$ years under table 2 is the square of that under table 1.

   (ii) Suppose that table 1 follows the Gompertz Law from the previous question. Show that the probability of survival for $n$ years for a life aged $x$ under table 2 is the same as that under table 1 for a life aged $x+a$, for some $a > 0$. Find $a$. Comment on the result.

4. (i) Show algebraically that $A_{x:\overline{n}|} = v\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n-1}|}$. Also demonstrate the result verbally.

   (ii) Show algebraically that $(IA)_x = \ddot{a}_x - d(I\ddot{a})_x$. Demonstrate the result verbally.
   *[Here $(IA)_x$ is the expected time-0 value (assuming an age of $x$ at time 0) of a single payment of size $K+1$ made at time $K+1$, where death occurs in the year $(K, K+1)$. $(I\ddot{a})_x$ is the expected value of the stream of payments of 1 at time 0, 2 at time 1, …, $K+1$ at time $K$, with the last payment made at the beginning of the year of death.]*

5. Suppose that the future lifetime of a life aged $x$ ($x > 0$) is represented by a random variable $T_x$ distributed on the interval $(0, \omega - x)$, where $\omega$ is some maximum age.

   (i) For $0 \le x < y < \omega$, state a consistency condition between the distributions of $T_x$ and of $T_y$.

(ii) Suppose that $0 \leq x < y < \omega$, and let $t = y - x$. Define the force of mortality at age $y$: (a) in terms of $T_0$; and (b) in terms of $T_x$, and show that the two definitions are equivalent.

(iii) Prove that ${}_tp_x = \exp\left\{-\int_0^t \mu_{x+s}ds\right\}$.

6. (a) Show that at age $x$ if $0 \leq a < b \leq 1$ then ${}_{b-a}q_{x+a} = 1 - \dfrac{{}_bp_x}{{}_ap_x}$.

(b) Hence, or otherwise, show that if deaths occurring in the year of age $(x, x+1)$ are uniformly distributed that year, then ${}_{b-a}q_{x+a} = \dfrac{(b-a)q_x}{1-aq_x}$.