

Problem Sheet 3 - Part B Actuarial Science II - Oxford HT 2006

Arbitrage-free pricing

1. There are two securities S^1 and S^2 in a market. The values of both securities at $t = 0$ is 1. The value of S^1 at $t = 1$ is $1 + i$. The value of S^2 at $t = 1$ is either u or d , each with positive probability. Assume $0 < d < u$. Show that an arbitrage opportunity exists in this model if and only if either $1 + i \geq u$ or $1 + i \leq d$.
2. Consider an idealised one-period securities market model consisting of two assets. At the end of the period, the market will be either “up” or “down”. The first asset pays £5 if the market is up and £1.25 if the market is down. The second asset pays £1.25 if the market is up and £5 if the market is down. At $t = 0$, the first asset sells for £2.75 and the second for £1.90. An investor would like to receive £1.25 if the market is up and nothing if the market is down. Find portfolio holdings of the two assets which have the payoff the investor would like. What is the current price of such a portfolio?
3. (a) Suppose that there exists a risk-free asset with constant force of interest r . Using the no arbitrage assumption, show from first principles that the forward price F , agreed at time 0 and to be paid at time T , for an asset S , with no income and with value S_0 at time 0, is given by $F = S_0 e^{rT}$. Assume that there are no transaction costs.
(b) Extend the argument in (a) to derive the forward price where the asset provides a fixed known (cash flow of) income.
4. The forward rate from time t to time $t + 1$, $f_{t,1}$, has the following values

$$f_{0,1} = 4.0\%, \quad f_{1,1} = 4.5\%, \quad f_{2,1} = 4.8\%.$$

- (a) Assuming no arbitrage, calculate
 - i. the price per £100 nominal of a 3-year bond paying an annual coupon in arrears of 5%, redeemed at par in exactly three years, and
 - ii. the gross redemption yield from the bond.
- (b) Explain why a bond with a higher coupon would have a lower gross redemption yield, for the same term to redemption.
5. An asset has a current price of $100p$. It will pay an income of $5p$ in 20 days' time. Given a risk-free rate of interest of 6% per annum convertible half-yearly and assuming no arbitrage, calculate the forward price to be paid in 40 days.
6. $f_{t,r}$ is the forward rate applicable over the period t to $t + r$. i_t is the spot rate over the period 0 to t . The gross redemption yield from a one-year bond with a 6% annual coupon is 6% per annum effective; the gross redemption yield from a

two-year bond with a 6% annual coupon is 6.3%; and the gross redemption yield from a three year bond with a 6% annual coupon is 6.6% per annum effective. All the bonds are redeemed at par and are exactly one year from the next coupon payment.

- (a)
 - i. Calculate i_1 , i_2 and i_3 assuming no arbitrage.
 - ii. Calculate $f_{0,1}$, $f_{1,1}$ and $f_{2,1}$ assuming no arbitrage.
 - (b) Explain why the forward rates increase more rapidly with term than the spot rates.
7. The current price of a security is $S_0 = \mathcal{L}2$. In 6 months' time the security will pay a dividend equal to $10p + 0.04S_{\frac{1}{2}}$ (where $S_{\frac{1}{2}}$ is the price of the security in 6 months' time). A riskless asset is available with constant interest rate 5% pa. What forward price should be agreed now to buy the security in 9 months' time?