Proof: $({}_tV_x + P_x) = (A_{x+t} - P_x\ddot{a}_{x+t}) + P_x$

but $A_{x+t} = v\,q_{x+t} + v\,p_{x+t}A_{x+t+1}$

and $\ddot{a}_{x+t} = 1 + v\,p_{x+t}\ddot{a}_{x+t+1}$

so $({}_tV_x + P_x) = v\,q_{x+t} + v\,p_{x+t}A_{x+t+1} - P_x(1 + v\,p_{x+t}\ddot{a}_{x+t+1}) + P_x$

$$= v(q_{x+t} + p_{x+t}(A_{x+t+1} - P_x\ddot{a}_{x+t+1}))$$

$$= v(q_{x+t} + p_{x+t} * {}_{t+1}V_x)$$

or $({}_tV_x + P_x)(1+i) = q_{x+t} + p_{x+t} * {}_{t+1}V_x$

This can also be seen from general reasoning.

The value held at the start of the year, plus the premium collected at the start of the year, accumulate with interest to provide at the end of the year sufficient to provide:

Death benefit (1 payable with probability q_{x+t})
Policy value at the start of the next year (with probability of survival p_{x+t})

Similar equations can be produced for other types of assurance contract.

Thiele's Equation

This is the equivalent for continuous functions.

We have ${}_t\bar{V}_x = \bar{A}_{x+t} - \bar{P}_x\bar{a}_{x+t} = 1 - \bar{a}_{x+t}/\bar{a}_x$

Differentiating $\bar{a}_{x+t}$ we have

$$\frac{\partial}{\partial t}\bar{a}_{x+t} = \frac{\partial}{\partial t}\int_0^\infty e^{-\delta s}\,{}_sp_{x+t}\,ds = \int_0^\infty e^{-\delta s}\frac{\partial}{\partial t}\,{}_sp_{x+t}\,ds$$

$$= \int_0^\infty e^{-\delta s}\,{}_sp_{x+t}(\mu_{x+t} - \mu_{x+t+s})\,ds = \mu_{x+t}\bar{a}_{x+t} - \bar{A}_{x+t}$$

$$\therefore \frac{\partial}{\partial t}\,{}_t\bar{V}_x = \frac{-\mu_{x+t}\bar{a}_{x+t} + \bar{A}_{x+t}}{\bar{a}_x} = -\mu_{x+t}(1 - {}_t\bar{V}_x) + \frac{1 - \delta\bar{a}_{x+t}}{\bar{a}_x}$$

$$= -(1 - {}_t\bar{V}_x)\mu_{x+t} + \delta\left(1 - \frac{\bar{a}_{x+t}}{\bar{a}_x}\right) - \delta + \frac{1}{\bar{a}_x}$$

$$= -(1 - {}_t\bar{V}_x)\mu_{x+t} + \delta\,{}_t\bar{V}_x + \frac{1 - \delta\bar{a}_x}{\bar{a}_x}$$

$$\therefore \frac{\partial}{\partial t}\,{}_t\bar{V}_x = -(1 - {}_t\bar{V}_x)\mu_{x+t} + \delta\,{}_t\bar{V}_x + \bar{P}_x$$