$$f_x(t) = \lim_{h\to 0} \frac{1}{h} \times \left(P[T_x \leq t+h] - P[T_x \leq t]\right)$$

$$= \lim_{h\to 0} \frac{1}{h} \times \left(P[T \leq x+t+h \mid T > x] - P[T \leq x+t \mid T > x]\right)$$

$$= \lim_{h\to 0} \frac{1}{S(x).h} \left(P[T\leq x+t+h] - P[T\leq x] - \left(P[T\leq x+t] - P[T \leq x]\right)\right)$$

$$= \lim_{h\to 0} \frac{1}{S(x).h} \left(P[T\leq x+t+h] - P[T\leq x+t]\right)$$

$$= \frac{S(x+t)}{S(x)} \lim_{h\to 0} \frac{1}{h} \frac{\left(P[T\leq x+t+h] - P[T\leq x+t]\right)}{S(x+t)}$$

$$= S_x(t) \left(\lim_{h\to 0} \frac{1}{h} P\left(T \leq x+t+h \mid T > x+t\right)\right.$$

$$= S_x(t) \times \mu_{x+t}$$

$$f_x(t) = {}_tp_x \ \mu_{x+t}$$

$${}_tq_x = F_x(t) = \int_0^t f_x(s)\,ds = \int_0^t {}_sp_x \ \mu_{x+s}\,ds$$

$$\frac{\partial}{\partial s}\,{}_sp_x = -\frac{\partial}{\partial s}\,{}_sq_x = -f_x(s) = -{}_sp_x\,\mu_{x+s}$$

Now $\frac{\partial}{\partial s} \log_s p_x = \dfrac{\frac{\partial}{\partial s}\,{}_sp_x}{{}_sp_x}$

$$\therefore \quad \frac{\partial}{\partial s} \log_s p_x = -\mu_{x+s}$$

$$\Longrightarrow \int_0^t \frac{\partial}{\partial s} \log_s p_x\,ds = -\int_0^t \mu_{x+s}\,ds + c \quad \text{(integration constant)}$$

$$\text{LHS} = \left[\log_s p_x\right]_0^t = \log_t p_x \qquad (\text{since } {}_0p_x = 1)$$

Taking exponentials

$${}_tp_x = \exp\left\{-\int_0^t \mu_{x+s}\,ds + c\right\}$$

Since ${}_0p_x = 1$ $\quad$ c must $= 0$

$$\therefore \quad {}_tp_x = \exp\left\{-\int^t \mu_{x+s}\,ds\right\}$$